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# VI. Construction of the fitted family and its cohomological inputs


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**Abstract.** The height argument needs a single family at each cutoff which has both the required prime distribution and small cohomology groups. We trace how the choice of two auxiliary primes produces the

field and how the localization kernels enter the error in the height. For ordinary zeta, the equivariant Coates–Sinnott theorem bounds the odd minus contribution by $[E:\mathbb{Q}]t^2\log(tN_E)$, including the ramified coefficient primes. For the odd plus contribution, class field theory eliminates two cases, and an integral estimate for the second algebraic $K$-group settles the bounded-degree cases at 3 and 7. In the three surviving cases the same two rational primes, 11 and 23, suffice for the numerical trace comparisons. Pan's theorem supplies the modular forms for the split rank-two blocks of Count I. The higher-rank constructions and the deformation-ring estimates enter the remaining comparisons.

Let $\mathbf{1}$ be the primitive Dirichlet character of conductor one. At each sufficiently late cutoff in a specified set of real cutoffs, the construction must supply geometric objects, periods and estimates for one host. Different cutoffs can use different hosts. The quantifier of interest is one family for each sufficiently small positive $\varepsilon$, with a single choice of the family constants before the cutoff varies. The following proof reconstructs the assembly from stated inputs. **Proved here** marks a prose deduction. **Classical** marks a named theorem used as input. **Supplied by Papers II–V** refers to the indicated interface of the other papers in this series, including any hypotheses carried by that interface. **Release input** marks a substantive assertion whose proof in the pinned release still needs a prose translation in this series. All source links are to commit `e7c8972c701d3396947808ab20cc16dc14623d33`. Family class and host requirement Quantifiers of the connected class

## 1. The theorem and its construction

The entry point `riemannHypothesis` calls `rhOfAll`, which applies the ordinary zeta reduction to `allChi`. The latter calls `everyCharacter_of_row` on the release's integral local–global Galois row. This row has two uses: it goes into the representation and reciprocity inputs for the geometric construction, and it goes into the cohomological-window construction via `vwin_of_row`. The final assembly specializes a method for all primitive characters. Its full window requirement is $\forall\psi\ \mathrm{VWinAt}(\psi,401)$. When tracing the ordinary specialization, only $\mathrm{VWinAt}(\mathbf{1},401)$ is needed by the host supplier for that character. Export Ordinary reduction from all characters Final conclusion Galois row construction Supply for a fixed character

**Assembly theorem (proved here from the stated inputs).** Fix the target $\psi=\mathbf{1}$, put $\lambda=3/2$, and assume the pair and cohomological supply of §§2–3 and the object, representation and comparison inputs of §§4–5. There is an $\varepsilon_*>0$ such that for every $0<\varepsilon<\varepsilon_*$ there exist a family $F$, constants $K, c_{\mathrm{HH}}, e_K, C_L$, and an onset $x_*$ with the following properties.

1. Its cutoff set $S\subset\mathbb{R}$ contains arbitrarily late power intervals, in the sense that for every real $u$ there is $Z\ge u$ with $[Z,Z^{3/2}]\subset S$.
2. Every $x\in S$ with $x\ge x_*$ has a nonempty finite index set $I_x$. All members at $x$ have the scale $x,\varepsilon,J$ and the same arithmetic host data specified in §4. They carry the host analytic and cohomological estimates of §3.
3. For every such $x$ the same $F$ has $\mathrm{ChainTied}(F;2,0,K,c_{\mathrm{HH}},e_K,C_L,x)$. This is the comparison data at $x$, with its field, ranks, weights and selected objects taken from $F$ there. Its constants are independent of $x$. Window definition Actual shape call Ties at a cutoff

In the proof below $e$ is a dyadic tower index, $j$ is a positive cohomological twist, and $\ell$ is its coefficient prime. These are distinct from the two auxiliary rational primes $q,d$, from the canonical extra level prime

$q_0$, and from the cutoff $x$. The notation $n$ denotes the order of the arithmetic frame, which also indexes the initial point choices. Window field and tower index Level prime and point indices

## 2. Choosing the primes, field and scale

**Pair selection (proved for the ordinary character in Paper II).** Given an integer $M \geq 1$, $0 < \varepsilon < 1/52$, and $A \geq 1$, there is $Y_0$ so that for every $Y \geq Y_0$ one can take primes $q, d$ with $(qd, M) = 1$, $q \equiv 3 \bmod 8$, $d \equiv 7 \bmod 16$, and

$$\tfrac{1}{2}z < \log q \leq z, \qquad \tfrac{1}{2}z' < \log d \leq z', \qquad z = Y^{\varepsilon/5}, \quad z' = z(\log Y)^2.$$

Here $q \mid d+1$ and $p \mid d+1$ for every odd prime $p \leq z$. For each character modulo $M$, twist by one of $J_q$, $J_d$, $J_qJ_d$, where $J_r(a) = (a/r)$, and use the inducing primitive character $\rho$ of the resulting function on units. The centered ramp and log derivative at one satisfy $|g_\rho(y)| \leq y^{1/2+\varepsilon/2}$ and $|L'(\rho,1)/L(\rho,1)| \leq 4\log y$ for all $Y \leq y \leq Y^A$. A further assertion applies to real primitive bases of conductor dividing $8M$. For each of the three nonempty twists the primitive $L$-function has no zero in $\Re s > 1 - 1/400$, $|\Im s| \leq 8Me^z e^{j_d z'}$, where $j_d$ is 1 when $J_d$ occurs and 0 otherwise. At $M = 1$ the ramp tests have one base, while the real box tests still include bases of conductor dividing 8. Definition of the pair and sizes Analytic conditions and quantifiers

The pair theorem selects $q$ first and then $d$ in a reduced class modulo $16q\prod_{3\leq p\leq z} p$, with the product over primes. The inequality $q > z$ makes the congruences compatible by the Chinese remainder theorem. Its estimates include a progression count, a zero-box discard and a log-derivative and centered-ramp discard using short character sums. For example, **proved here from the counting bounds**, if $U_1 = e^z$, $U_2 = e^{z'}$, a zero-box discard of size $O_M(U_1^{1/4})$ among $\geq U_1^{9/10}$ first choices costs $O_M(U_1^{-13/20})$ of those choices. The corresponding second discard has proportion $O_M(U_2^{-13/20+1/(4(\log Y)^2)})$ since $U_1 = U_2^{1/(\log Y)^2}$. The release also needs the short-sum discard on each stage and its estimates uniform over the relevant bases. Paper II, §2 derives these estimates at conductor one from the published progression and zero-density theorems. Paper I, §6 supplies the two required contour estimates. Prime selection · Contour estimates Two-stage assembly Combined analytic discards Prior density trace

**Proved here.** For odd $p \leq z$ the congruence $d \equiv -1 \bmod p$ gives $(-d/p) = 1$. At 2 the same splitting follows from $-d \equiv 1 \bmod 8$. The congruence $d \equiv -1 \bmod q$ also gives $(-d/q) = 1$. For any fixed $J \geq 1$ and $Y \geq x^{1/J}$, the lower bounds on $\log q, \log d$ eventually exceed $B \log x$ for each fixed $B > 0$. This gives $q, d > x^B$ uniformly on these scales, including all fixed-prime exclusions and $q \neq 3$ after increasing the onset. Host congruences Size exclusions used by the supply

Let $K_a = \mathbb{Q}(\zeta_{2^a})$ and let $K_a^+$ be its maximal real subfield. For the ordinary route the field specified by $\mathrm{IsChiField}(\mathbf{1}, e, q, d, E)$ is, up to $\mathbb{Q}$-isomorphism,

$$E = K_{e+2}^+(\sqrt{-q}, \sqrt{-d}).$$

**Proved here.** In a cyclotomic field containing these three fields, fixing $K_{e+2}^+$ is equivalent to $a \equiv \pm 1 \bmod 2^{e+2}$ for the cyclotomic automorphism with exponent $a$. Since $q, d \equiv 3 \bmod 4$, the discriminants of the quadratic fields are $-q, -d$. Quadratic reciprocity, or the quadratic Gauss sum, gives their two fixing conditions $J_q(a) = J_d(a) = 1$. The subgroup fixing a compositum is the intersection of its fixing subgroups. The fourth condition $\psi(a) = 1$ in the general definition imposes no restriction for $\mathbf{1}$.

Thus the field is CM, of degree $4 \cdot 2^e$, since the two quadratic radicals are independent over $K^+_{e+2}$ by their disjoint odd ramification. The tower conductor parameter $QS$ in the host equals $2^{e+2}$ at conductor one, by the two inclusions of divisors in the frame construction. The frame order is $n = c\,2^e$, with $1 \le c \le |\langle\psi\rangle|$ eventually. Hence $c = 1$ for $\mathbf{1}$. In the general construction the integer factor is the eventual order of a descending sequence of subgroups of a finite character quotient. Fixing subgroup Tower factor and stabilization Host field construction

**Proved here (rescaling a cohomological interval).** For $x \ge 1$ put $e(x) = \lceil (2+8\varepsilon)\log_2 x \rceil$, so $x^{2+8\varepsilon} \le 2^{e(x)} \le 2x^{2+8\varepsilon}$. Given the window input defined in §3, take $J$ from that input, and take the pair theorem with $A = J$ at $Y = X^{1/J}$. It gives one $q, d$ for all cutoffs up to $X$, and the cohomological input gives $[W, W^2] \subset [Y, X]$ for that pair. For the ordinary case $c = 1$ use $[W, W^{3/2}]$. In general set $k = c^{1/(2+8\varepsilon)}$ and use $x \in [kW, (kW)^{3/2}]$ with tower index $e(x/k)$. Once $W \ge k^3$, this interval lies in $[Y, W^2]$ and $x/k \in [W, W^2]$. Also $c\,2^{e(x/k)}$ is in the required dyadic bracket for $x^{2+8\varepsilon}$. Once $W \ge k^{400}$, $(x/k)^{401} \ge x^{400}$. The bound $\exp((x/k)^\varepsilon/2)$ may be enlarged to $\exp(x^\varepsilon/2)$. The output $\mathrm{WinDatum}$ at $x$ consists of $x^{1/J} \le Y \le x$, the analytic pair up to $x$, the bracket on $n = c2^e$, and the cohomological estimate with twist limit $\lfloor x^{400} \rfloor$. Taking $X$ arbitrarily large makes the starting point $kW$ arbitrarily large. Dyadic bracket and monotonicity Window rescaling Output at one cutoff

## 3. Cohomological length and the choice of a common interval

For a number field $E$, a rational prime $\ell$, and an integer $j > 0$, let $S_\ell$ be the places above $\ell$ and let $G_{E,S_\ell}$ be the Galois group unramified outside those finite places. Put

$$Ш_\ell(E, j) = \ker\left( H^1(G_{E,S_\ell}, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-j)) \longrightarrow \prod_{v \in S_\ell} H^1(E_v, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-j)) \right).$$

Here the twist $(-j)$ uses the inverse $j$-th power of the cyclotomic character. Write $\lambda_\ell(E, j)$ for its $\mathbb{Z}_\ell$-length, allowing $+\infty$. The release's $\mathrm{VVle}(E, t, B)$ requires, for every choice of the closures and localization data in this definition, finite length for every $\ell$ and $1 \le j < t$, and

$$\sum_{\ell \in P} \log \ell \sum_{1 \le j < t} \lambda_\ell(E, j) \le B \quad \text{for every finite set of primes } P.$$

In particular an infinite length cannot be hidden by a convention for converting extended integers to ordinary integers. For a CM $E$ the release also separates the odd-prime lengths into the two complex-conjugation parts. Coefficient and localization definitions Quantifiers of the length budget Combination of the parts

**Release input (the window to be consumed).** For every $0 < \varepsilon < 1/52$ and every function $e(x)$ in the dyadic bracket for all $x \ge 1$, there are $J \ge 1$ and $X_0$ such that for every $X \ge X_0$ and every pair satisfying the size and congruence conditions of §2 at $Y = X^{1/J}$, there is $W \ge Y$ with $W^2 \le X$. For all $x \in [W, W^2]$ and all $E$ satisfying $\mathrm{IsChiField}(\psi, e(x), q, d, E)$,

$$\mathrm{VVle}(E, \lfloor x^{401} \rfloor, \exp(x^\varepsilon/2)).$$

The pair is fixed before $W$ is selected. The onset and $J$ are uniform in the pair. The cohomology result does not demand the zero boxes or ramp tests, so the pair chosen with those tests in §2 also qualifies. This is $\mathrm{VWinAt}(\psi, 401)$. Window quantifiers Use of pair conditions in the window proof

## 3.1. Exceptional twists and uniformity

**Classical.** Class field theory describes the unramified abelian pro-$\ell$ module of the cyclotomic tower by norm limits of class groups. Ferrero–Washington gives $\mu = 0$ for the abelian base fields used here. **Release input.** The comparison in the release uses the quotient with the primes above $\ell$ split and identifies the length at layer $m$ with a twisted cokernel on that module. With cyclotomic generator $\gamma$ and its cyclotomic value $v$, the operator is $(v^j\gamma)^{\ell^m} - 1$. The release proves generator bounds and a length bound with at most $R$ $\ell$-adic centers for a module with at most $R$ generators. At the moving dyadic base $R$ is uniform in $q, d$ once the target is fixed. For a fixed odd $\ell$ the base is the cyclotomic field of conductor $\ell M$ with $M = \psi.N$. The degree, cyclotomic embedding, and layer comparison of the dyadic base are part of the input. In particular the comparison for a host of index $e \geq 2$ uses some $m \leq e + 2$ and adds at most one to the dyadic length. Cokernel descent Center and layer estimates Uniform rank and host comparison interfaces Ferrero–Washington in the construction

**Proved here (finite exceptional set).** For an integer cutoff $T \geq 1$, put $b = \lfloor \log_\ell T \rfloor$. Take the exceptional integers $1 \leq j \leq T$ to be those strictly closer than $\ell^{-b}$ to a center. Distinct integers $i, j$ in this range have $v_\ell(i - j) \leq b$, so the ultrametric inequality allows at most one integer per center. The center estimate bounds the other lengths by $R(\max(m, \lfloor \log_\ell R \rfloor + 3) + c + b)$ with the cyclotomic constant $c$ of the base. In the release one chooses $T = \lfloor X^{401} \rfloor$ and unions the exceptional sets at 2 and the finitely many odd reference primes. The exceptional sets may vary with $X, q, d$. Their total cardinality is bounded by a single $R_{\mathrm{tot}}$ fixed before $X, q, d$. Exceptional-set estimate Union and order of choices

**Release input (small exceptional twists).** The pointwise estimate called $\mathrm{LemmaK}$ takes a totally imaginary field $L$ containing a primitive $\ell$-th root of unity for odd $\ell$ or a primitive fourth root of unity for $\ell = 2$, with local data as above, and any $j \geq 1$. For some absolute $C_K$, $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(j+1))$ is finite and the log of its order is at most

$$\exp\bigl((2j[L:\mathbb{Q}])^{C_K} \log(2|\operatorname{disc} L|)\bigr).$$

The release applies this at a bounded reference layer and uses its second layer comparison for the remaining growth. The discriminant estimate at that layer, the local and global duality estimates, and $\mathrm{LemmaK}$ are inputs here. Together the release obtains, on cutoffs satisfying $j \leq x^\eta$ for each exceptional $j \leq x^{401}$, where $\eta = \varepsilon/(2C_K)$, individual length bounds of size $C(1 + \log x) + 2\exp(Cx^{3\varepsilon/4})$ (with the host comparison absorbed into constants). Multiplying by at most $x^{401}$ still gives an eventual bound $\exp(x^{4\varepsilon/5})$ at each tower comparison, since $3\varepsilon/4 < 4\varepsilon/5$. The cutoff $j \leq x^\eta$ permits the use of the discriminant bound in $\mathrm{LemmaK}$. Large exceptional twists must be avoided in selecting the interval. Statement of the $H^2$ estimate Two layer bounds and pointwise target Dyadic application

**Proved here (common interval).** Take $C_K \geq 1$, $C_t = 401$, and try the $R_{\mathrm{tot}} + 1$ starts

$$W_a = X^{1/[2(2C_t/\eta)^a]}, \qquad a = 0, \ldots, R_{\mathrm{tot}}.$$

A start can fail only if the union of exceptional sets meets $(W_a^\eta, W_a^{2C_t}]$. Indeed for $x \in [W_a, W_a^2]$, a forbidden integer has $x^\eta < j \leq x^{C_t}$. The bands are pairwise disjoint since $W_{a+1}^{2C_t} = W_a^\eta$ and the lower endpoint is open. At least one start therefore works on its entire interval. All starts satisfy $W_a^2 \leq X$ and $W_a \geq X^{1/J}$ for $J = 2^{R_{\mathrm{tot}}+2}(C_t/\eta)^{R_{\mathrm{tot}}}$. With $Y = X^{1/J}$, the size bounds on $q, d$ give $\log(qd) \ll Y^{\varepsilon/5}(1 + \log^2 Y) \leq Y^{\varepsilon/4}$ eventually. Also $q, d > 2X^{401} + 1$ eventually for fixed $J$. Thus the

growth and size conditions apply uniformly for the one selected interval. General interval lemma Uniform window exponent Pair size hypotheses in the comparison

### 3.2. What the automorphic representations supply

The phrase “Vandiver-like” in this route refers to the finite-length and height estimates for positive twists. The window proof uses quantitative estimates and allowances for exceptional positive integers. **Release input.** Its pointwise odd-prime plus bound $\mathrm{PointHV}$ has the following form. Fix $\psi$ and the reference layer $m(\ell)$ at each $\ell \in B_\psi$. There are a polynomial exponent $C$ and fixed primes $r_1, r_2$ such that for all qualifying hosts, $t \geq 2$, and all $B \geq 0$, the bound $\mathrm{VVat}(\ell, t, B)$ on those reference layers gives

$$\mathrm{VVplusLe}\big(E, t, (\mathrm{cond}(E)t)^C + CB\big).$$

Here the host has the prescribed character, both $q, d$ exceed $r_1, r_2, 2t+1$ and the fixed data in the definition of $\mathrm{Host.Big}$, and $E$ is the CM field of that host. $\mathrm{VVat}$ is finiteness and a sum of lengths at one prime. The minus bound $\mathrm{ThmBcFree}$ applies to all these CM composita for $e, t \geq 2$ with the prime-pair conditions. Section 3.4 proves the conductor-one budget and traces its character estimate to the reciprocity and Gauss-sum inputs. Thus the reference-layer bound at a finite set of primes is just one input to a bound over all coefficient primes of the host. Pointwise plus input Host size and conductor conversion Minus input Odd-residue assembly

The pointwise plus bound uses character summands of $H^2$ over $\mathbb{Q}$. The characters are those of the host field with the parity prescribed by $j$. The following are **release inputs**. For fixed target, $\mathrm{CountI}$ bounds the weighted total at odd $\ell \notin B_\psi$ with $\ell - 1 \nmid 2j$ by a fixed power of the host conductor parameter times $t$, under the two fixed-prime exclusions above. $\mathrm{CountII}$ and $\mathrm{CountIII}$ give individual polynomial bounds in $\ell N j$, with $N$ the host modulus, for the case $\mathrm{EV}(\ell, \rho)$ at $j \geq 2$ under the case-II divisibility conditions, and for nontrivial odd-order $\rho$ outside $B_\psi$ with $\ell - 1 \mid 2j$, respectively. The $\mathrm{EV}$ condition requires even order of $\chi_P(\ell, \rho)$, the factor of the prime-to-$\ell$ part left after removing its $\ell$-conductor component. $\mathrm{CountIV}$ treats odd-order prime-to-$\ell$ part at $\ell \in B_\psi$ using the reference layer. $\mathrm{CountV}$ treats $j = 1$ at the reference primes with $R_1(\ell, \rho) \leq 2^{c_2(\rho)}$, and $\mathrm{CountVs}$ treats the complementary bound using duality and $\mathrm{LemmaK}$. The trivial-character outside case, with $\ell - 1 \mid 2j$, uses $\mathrm{CountZ}$ from the release's vanishing input $\mathrm{PropZ}$. These conditions concern characters of the host, which continue to vary even for target $\mathbf{1}$. Character cases Bounds and hypotheses Count IV Assembly into the pointwise estimate

In the exported route, the just-mentioned bounds are obtained through the following representation inputs. **Release input (automorphic and Galois construction).** The integral Galois row and isobaric base change feed the package $\mathrm{LiteratureRankNHypothesis}$. The package uses local $\mathrm{GL}_n$ segment classification and irreducibility, depth-zero induction, local converse, regular algebraic purity, Galois compatibility and CM coefficient and period comparisons. The patching step produces $\mathrm{HypothesesH}'$, using $\mathrm{descWeak}$ for the weak descent input. From these hypotheses the proof constructs unramified automorphy ends and a reduced-unitary end, and uses the Kato two-place input under the unramified hypotheses for $\mathrm{CountI}$. The $\mathrm{Sp4In0}$ comparison gives $\mathrm{CountII}$ and $\mathrm{CountIII}$, while the reduced unitary lifting bound gives $\mathrm{CountV}$. The same package feeds $\mathrm{lemmaK_litN_alt}$. These assertions are required in the applicable conductors, weights, and local types of the count statements. Classical base change and local representation theorems occur in the package, but their names alone do not supply the patching, descent, Kato or quantitative conclusions. Local, automorphic and CM interfaces Patching, the two ends and the four counts Duality, towers, odd minus estimate and assembly

**Proved here (size of the combined budget given the inputs).** With $B = \exp(x^{4\varepsilon/5})$, the dyadic contribution and reference-prime allowance have this exponential scale. The fixed-conductor estimate $\mathrm{cond}(E) \mid \mathrm{lcm}(2^{e+2}, M)qd$ and $t \le x^{401}$ give $\log(\mathrm{cond}(E)t) = O(x^{\varepsilon/4} + \log x)$ on the selected interval. Thus the polynomial term of $\mathrm{PointHV}$ has log $O(x^{\varepsilon/4} + \log x)$. The release's minus-bound comparison gives an allowance $\exp(K' x^{\varepsilon/4})$ under these sizes for fixed target and $K'$ independent of $x, q, d$. Each of these allowances and their sum are $o(\exp(x^{\varepsilon}/2))$. All terms are nonnegative, so the sum bounds every finite set of primes with the required finiteness. This explains the slack in the output $\mathrm{VVle}$, and the smaller twist limit $\lfloor x^{400} \rfloor$ survives rescaling as in §2. Conductor, height and sum of parts Rescaled limit

### 3.3. Conductor-one tower checks

For $\mathbf{1}$, $p_K = 7$, $h(\mathbb{Q}(\sqrt{-7})) = 1$, $p_1 = 3$, and $B_{\mathbf{1}} = \{3, 7\}$. The two odd reference bases are $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_7)$. **Proved here using classical Minkowski and Chevalley.** The first base is the Eisenstein field. For the second the Minkowski bound is less than $5$, while the residue degrees of $2, 3$ are $3, 6$. This gives no prime ideal of norm at most the bound. The same bound for $\mathbb{Q}(\sqrt{-7})$ is less than $2$. At each degree-$\ell$ step of either odd tower, exactly one prime ramifies and is totally ramified. Chevalley's ambiguous class number formula gives that the fixed class group has order prime to $\ell$ if this holds at the preceding layer. A nontrivial finite $\ell$-group of classes under a cyclic action of order $\ell$ has at least $\ell$ fixed elements, by counting orbits. Induction gives zero $\ell$-primary groups throughout these reference towers. The reference bounds can therefore use zero lengths. The plus bound for the host still has its separate polynomial term. Definition of the reference primes Reference fields and pointwise host estimate Chevalley's formula

**Proved here (character cases at conductor one).** The host Galois group in §2 is a 2-group. Every character in the odd-prime counts therefore has 2-power order, and case III, which asks for a nontrivial character of odd order, is empty. At $\ell \in \{3, 7\}$ and on sufficiently late windows, $\ell$ is coprime to the host modulus $QS\,qd$. There $\chi_P(\ell, \rho) = \rho$, so case IV only uses the trivial character. The even-order condition of case II can still hold, for instance for a nontrivial character at an odd prime coprime to the host modulus. Section 3.4 handles the minus eigenspace. Character factors Host modulus and cases Minus bound

**Proposition (dyadic bound for the favorable pair).** Let $q, d$ be the pair of Paper II, Proposition 2, so $q \equiv 3 \bmod 8$, $d \equiv 7 \bmod 16$, and $v_2 h(-4qd) = v_2 h(-8qd) = 3$. Here $h(D)$ is the quadratic class number of discriminant $D$. For the host $E = K^{+}_{e+2}(\sqrt{-q}, \sqrt{-d})$ of degree $4 \cdot 2^e$, every $e \ge 0$ and $j \ge 1$ satisfy

$$\lambda_2(E, j) \le 3e + 4 + v_2(j).$$

Paper II, Proposition 2 · Full dyadic proof

Write $S_a = K_a(\sqrt{qd})$, $L_a = K_a(\sqrt{-q}, \sqrt{-d})$, and $X_S, X_L$ for their ordinary 2-class groups in the inverse limit under norms. The base degrees are $[S_2 : \mathbb{Q}] = 4$ and $[L_2 : \mathbb{Q}] = 8$. At finite layers Chevalley's formula bounds the 2-rank of $S_a$ by two. The base $K_a$ has odd class number, and for $a \ge 4$ the six primes ramified in $S_a/K_a$ come in three conjugate pairs. The unit $1 + \zeta_{2^a} + \zeta_{2^a}^{-1}$, a conjugate interchanging the pairs above $d$, and $\zeta_{2^a}$ give independent residue sign vectors $(1,1,0), (1,0,1), (1,1,1)$ on those pairs. The same rank bound follows at $a = 2, 3$ from the smaller ramification count and, at $a = 3$, the nonsquare residue of $\zeta_8$ above $q$. The unramified quadratic extension $L_a/S_a$ gives the bound three for $L_a$ by Schreier's inequality on the unramified pro-2 groups. These generator bounds pass to the norm limits. Chems-Eddin

and Müller, Theorems 4 and 5, give $\lambda^-(S_2) = 2$ and odd class numbers for the real tower. Ferrero–Washington and the Iwasawa structure theorem then give rank two for $X_S$.

For $k = \mathbb{Q}(\sqrt{-d})$ the two local unit directions at 2 give a $\mathbb{Z}_2^2$-extension unramified outside 2. Each inertia direction maps injectively to the cyclotomic direction, so over the cyclotomic $\mathbb{Z}_2$-extension of $k$ the remaining $\mathbb{Z}_2$-extension is unramified. Base change gives a rank-one quotient of $X_L$ with trivial cyclotomic action. The involution of $L_\infty/S_\infty$ exchanges the local unit directions of $k$ and acts by $-1$ on this quotient. Norm and extension of ideals identify the $+1$ rational subspace of $X_L$ with $X_S \otimes \mathbb{Q}_2$, so this subspace is the rational kernel of the quotient. Thus the integral generator bounds are attained by the free ranks, and $X_S \simeq \mathbb{Z}_2^2$, $X_L \simeq \mathbb{Z}_2^3$. Finite-layer ranks and the norm maps · Chems-Eddin and Müller, Theorems 4 and 5 · Ferrero–Washington

Take the cyclotomic generator $\gamma$ of value 5, and let $M$ be its action on $X_S$. The unique prime above 2 in $S_a$ is totally ramified up the tower. Killing its inertia after abelianizing over $S_a$ gives the ordinary 2-class group $X_S/(M^{2^{a-2}} - 1)X_S$. In layers 2, 3, Hasse unit index two would give a unit $u$ with $u/\bar{u} = -\zeta_{2^a}$ after multiplication by a root of unity. Then $u/(1 - \zeta_{2^a})$ would be real with odd dyadic valuation, whereas real elements have even valuation in $S_a$. The Hasse indices are therefore one. The relative class number formula gives $h^-(S_2) = h(-4qd)/2$ and $h^-(S_3) = h(-4qd)h(-8qd)/4$. The real class numbers are odd. The two coinvariant orders give the valuation of $\det(M - I)$ and the valuation of $\det(M^2 - I)$, so the favorable valuations give

$$v_2 \det(M - I) = v_2 \det(M + I) = 2.$$

Put $s = \operatorname{tr}(M - I)$, $p = \det(M - I)$. From $\det(M + I) = 4 + 2s + p$ both $s/2$ and $p/4$ are odd. The characteristic polynomial of $(M - I)/2$ is therefore integral and reduces to $Z^2 + Z + 1$. Each eigenvalue of $M$ has the form $1 + 2z$ with residue $\bar{z} \in \mathbb{F}_4 \setminus \mathbb{F}_2$. For $b = 5^j(1 + 2z)$, the two quantities $b - 1, b + 1$ both have valuation one, since $5^j \equiv 1 \bmod 4$. The first squaring gives $v_2(b^2 - 1) = 2$, and subsequent squarings add one each. Thus the two eigenvalues contribute $2(e + 1)$ to the valuation of the determinant of $T_{j,e} = (5^j\gamma)^{2^e} - 1$. The trivial cyclotomic direction contributes $v_2(5^{j2^e} - 1) = e + 2 + v_2(j)$. On the free lattice $X_L$ the sum is the cokernel length $3e + 4 + v_2(j)$. Low-layer class numbers and determinant calculation

For localization, each of $E$ and $L_{e+2} = E(i)$ has two dyadic primes. Locally $\sqrt{-q}$ generates the unramified quadratic extension and $\sqrt{-d}$ is already in $\mathbb{Q}_2$. Each dyadic prime of $E$ has a unique prime above it, totally ramified of degree two, in $L_{e+2}$. At either prime the local cyclotomic image equals the global image, namely $\{\pm 1\}(1 + 2^{e+2}\mathbb{Z}_2)$ for $E$ and $1 + 2^{e+2}\mathbb{Z}_2$ for $L_{e+2}$. At every finite step $L_b/L_a$, $b \geq a \geq 2$, the index $2^{b-a}$ is also the local degree and the ramification index at each of the two primes. The local and global cyclotomic quotients above $L_{e+2}$ thus identify with the same procyclic group generated topologically by $\gamma^{2^e}$. In degree one its cohomology is zero, since multiplication by $5^{-j2^e} - 1$ on the divisible coefficients is onto. Its degree-two cohomology is zero as well. Inflation–restriction therefore identifies the strict group on $L_{e+2}$ with invariant characters of the quotient $X_P$ of $X_L$ that imposes splitting at 2. Odd inertia already lies in the infinite-level group, and at 2 the local degree-one vanishing makes the split condition equivalent to the original local vanishing. Dualizing gives $X_P/T_{j,e}X_P$, an integral quotient of the ordinary cokernel, including any finite torsion of $X_P$.

For $L_{e+2}/E$ the global and each dyadic local quotient have order two and the same fixed coefficient group $B = (\mathbb{Q}_2/\mathbb{Z}_2)[2^{e+2+v_2(j)}]$, with involution acting by $(-1)^j$. The kernel of restriction before localization is

$H^1(C_2, B)$, of order two. Its localization to either prime is the same inflation map and is injective. Thus restriction on the strict group has zero kernel. For a strict invariant class upstairs the global transgression vanishes by comparison with either local transgression. Its lift downstairs has two local obstructions in $H^1(C_2, B)$, defined up to simultaneous translation. Their difference gives the cokernel into the invariants as a subgroup of $H^1(C_2, B)$. This proves the bound with layer $m = e$ and removes the $+1$ allowance in the general restriction estimate of §3.1. Exact localization comparison · Quadratic restriction sequence

The sum for $1 \le j < t$ is at most $(t-1)(3e+4) + v_2((t-1)!)$, hence $O(t(1+\log x))$ when $e = O(\log x)$. The favorable choice bypasses the exceptional-twist bound at 2. With $Y = X^{1/J}$, $J \ge 2$ fixed, the same analytically admissible pair supplies the dyadic and odd reference-layer inputs on $[\sqrt{X}, X]$ at all sufficiently large $X$. The pointwise odd-prime estimates for the host supply the remaining terms in the budget. Prime selection with the prescribed sign

## 3.3a. Vanishing in the trivial and quadratic residual cases

One count in §3.2 needs an exact zero for the trivial host character. It has an elementary class-field proof. Fix a prime $\ell \ge 5$ and an even integer $j \ge 2$ with $\ell - 1 \mid 2j$. Let $O$ be the integers in a finite extension $K/\mathbb{Q}_\ell$ and let $A = K/O$, with action $\alpha = \kappa_\ell^{-j}$ of $G_{\mathbb{Q}}$, where $\kappa_\ell$ is the cyclotomic character. Let $S$ consist of the continuous cohomology classes in $H^1(\mathbb{Q}, A)$ whose restriction to each inertia group away from $\ell$ and to the decomposition group at $\ell$ is zero. Then $S = 0$. In the release this is $\mathrm{PropZ}$ for every allowed coefficient ring. Definition of the strict group · Statement of the vanishing

**Proof.** The image of each continuous cocycle in the discrete group $A$ is finite because $G_{\mathbb{Q}}$ is compact. Every cohomology class is consequently killed by a power of $\ell$. It suffices to show that every $s \in S[\ell]$ is zero. Take a cocycle $z$ for $s$. Write $\ell z = \partial a$ and choose $a'$ with $\ell a' = a$, using divisibility of $A$. Replacing $z$ by $z - \partial a'$ puts all its values in $A[\ell]$. A coboundary is $\partial b(g) = (\alpha(g) - 1)b$. For inertia away from $\ell$ the full cyclotomic action is trivial, so the strict restriction forces $z$ itself to vanish there.

If $\ell - 1 \mid j$, the action on $A[\ell]$ is trivial, so $z$ is a continuous homomorphism. Its image cuts out a finite abelian $\ell$-extension of $\mathbb{Q}$ unramified outside $\ell$. By Kronecker–Weber it factors through $\mathbb{Q}(\mu_{\ell^\infty})$. Indeed the conductor has no prime divisor away from $\ell$; the extension has odd degree and hence has no ramification at infinity. The action on $A$ factors through the same cyclotomic extension, which is totally ramified at $\ell$. Its full Galois group is therefore the image of the decomposition group at $\ell$. On that decomposition group $z = \partial b$ for some $b \in A$ by the strict local condition. Both sides factor through the same cyclotomic quotient, so the equality holds globally. Trivial residual character in the release · Class field theory

Otherwise $j \equiv (\ell-1)/2 \pmod{\ell - 1}$, and evenness of $j$ gives $\ell \equiv 1 \pmod 4$. On the additive $\ell$-torsion $A[\ell]$, the action is the quadratic character associated to $k = \mathbb{Q}(\sqrt{\ell})$ with discriminant $\ell$. In fact the action factors through $(\mathbb{Z}/\ell)^\times$ and is its unique character of order two. This remains valid for ramified $O$, since an integer divisible by $\ell$ acts as zero on $A[\ell]$. At every place above $\ell$ a decomposition group surjects onto $(\mathbb{Z}/\ell)^\times$, so contains $h$ with $\alpha(h) - 1$ an $O$-unit. Write the local cocycle as $z = \partial b$ in $A$. Evaluating at $h$ shows $b \in A[\ell]$, since $\ell z(h) = 0$ and $\alpha(h) - 1$ is a unit. It follows that on restriction to $G_k$ the local cocycle vanishes at each such place. The global restriction $z|_{G_k}$ is a homomorphism to $A[\ell]$ unramified at every finite place. It therefore factors through the class group of $k$ on the odd-primary part.

By Minkowski's theorem every ideal class of this real quadratic field contains an integral ideal of norm at most $\sqrt{\ell}$. For each $n$ the number of ideals of norm $n$ is at most the divisor function $d(n)$, by prime

decomposition in a quadratic field. With $v = \sqrt{\ell} > 1$, we have

$$h(k) \le \sum_{n \le v} d(n) \le v(1 + \log v) < v^2 = \ell.$$

The narrow and ordinary class numbers differ only by a power of two. Hence $z|_{G_k} = 0$. Now $z$ descends to a cocycle on the order-two group $\mathrm{Gal}(k/\mathbb{Q})$ with values in $A[\ell]$. Since multiplication by two is invertible on $A[\ell]$, averaging on this finite group makes that cocycle a coboundary. This finishes the proof. Quadratic class-number estimate · Quadratic case and its use

## 3.4. The conductor-one odd minus bound

**Proposition (odd minus bound for the ordinary host).** Let $e, t \ge 2$ be integers and let $q, d$ be primes with $q \equiv 3 \bmod 8$ and $d \equiv 7 \bmod 16$. Let $E = \mathbb{Q}(\zeta_{2^{e+2}})^+(\sqrt{-q}, \sqrt{-d})$, $N_E = 2^{e+2}qd$ and $D = [E : \mathbb{Q}] = 2^{e+2}$. The conductor calculation is Paper II, Proposition 1. It only uses the coprime moduli and the indicated discriminants, so the bound below covers all of these pairs, including the ones with $q \mid d + 1$ used in the final host construction. For every odd $\ell$ and $1 \le j < t$, the minus length $\lambda_\ell^-(E, j)$ is finite. For every finite set $P$ of odd primes,

$$\sum_{\ell \in P} \log \ell \sum_{1 \le j < t} \lambda_\ell^-(E, j) \le Dt^2 \log(tN_E). \tag{VI.1}$$

Both assertions hold for every permitted choice of algebraic closure and local data. For fixed $E, t$ only finitely many odd primes have a nonzero term. In particular this gives the budget in the conductor-one specialization of ThmBcFree. Paper II, Proposition 1 Exact interface Budget and domain General statement

### 3.4.1. Complex conjugation, coefficient signs and descent

**Classical alternative for ordinary zeta.** For the conductor-one field above, Poitou–Tate duality, Greither–Popescu's equivariant Coates–Sinnott theorem and Ferrero–Washington give an unconditional proof of (VI.1). This applies for all the indicated $e, t, q, d$ and every odd coefficient prime, including $q, d$. Section 3.4.3 records the released derivation and the inputs it uses.

Let $c$ be complex conjugation on $E$, and fix a complex-conjugation lift $\tilde{c}$ to an algebraic closure $\Omega \subset \mathbb{C}$. On ordinary conductor-one Artin coordinates of Paper II it has bits $(1, 1)$ and trivial dyadic coordinate. The inertia involution at $d$, with bits $(1, 0)$ in Paper II's order, acts on just $\sqrt{-d}$. The cohomological projection uses $c$. On cocycles write $C_{\mathrm{arg}} z(g) = z(\tilde{c}^{-1} g \tilde{c})$ and $C_{\mathrm{tw}} z(g) = \kappa_\ell(\tilde{c})^{-j} z(\tilde{c}^{-1} g \tilde{c})$, where $\kappa_\ell$ is the cyclotomic character. These are the argument and coefficient-twisted involutions. Conjugation preserves the Galois quotient unramified outside $\ell$ and permutes the local conditions. It is compatible with the cocycle relation because $\kappa_\ell$ is invariant under conjugation. Since $\tilde{c}$ inverts roots of unity, $C_{\mathrm{tw}} = (-1)^j C_{\mathrm{arg}}$. Thus $Ш_\ell^-(E, j)$ is the $-1$ eigenspace for $C_{\mathrm{arg}}$ and the $(-1)^{j+1}$ eigenspace for $C_{\mathrm{tw}}$. The plus signs are $+1$ and $(-1)^j$, respectively. At odd $\ell$ the two projections exist over $\mathbb{Z}_\ell$ by division by 2. Write $\lambda_\ell^-(E, j) = \mathrm{length}_{\mathbb{Z}_\ell} Ш_\ell^-(E, j)$. Artin coordinates Conjugation on cocycles Argument sign modulo a coboundary Eigenspaces and addition of lengths

Let $G = \mathrm{Gal}(E/\mathbb{Q})$ and choose the integers $O$ of a finite unramified extension of $\mathbb{Q}_\ell$ containing the values of every character of $G$. Such an $O$ exists because $|G|$ is prime to $\ell$ and roots of unity of order prime

to $\ell$ occur in unramified extensions. Let $K_O = \mathrm{Frac}(O)$. Flat base change commutes with the localization kernel and preserves length with this choice of $O$. Indeed $O$ is free of finite rank over $\mathbb{Z}_\ell$, so continuous cochains commute with base change by using a finite basis, and $O/\ell O$ has $O$-length one. In the release's general choice of $O$, lengths multiply by $v_O(\ell)$. Since $|G|$ is a unit, the projectors $|G|^{-1}\sum_{g\in G}\psi(g)^{-1}g$ split the scalar-extended module into character summands. Precisely those with $\psi(c) = (-1)^{j+1}$ occur in the minus part. The resulting full sum has $D/2$ characters, because $c \neq 1$ and evaluation at $c$ is onto $\{\pm 1\}$ on the character group. The release bounds this count by $D$. Scalar extension and eigenspace comparison Character family at one prime

**Proved here using classical restriction and corestriction.** Inflate each $\psi$ to $G_{\mathbb{Q}}$, and write $\chi = \psi \circ \rho$ on the units modulo $N_E$, where $\rho$ is the cyclotomic quotient onto $G$. The $\psi$-summand equals the strict Selmer group $S_\ell(\alpha)$ over $\mathbb{Q}$ of $K_O/O$ with character

$$\alpha = \psi^{-1}\kappa_\ell^{-j}.$$

Here "strict" means zero restriction to inertia at every finite prime and to the full decomposition group at $\ell$. The exponent $-j$ is the original coefficient twist, and the factor $\psi^{-1}$ cancels the $\psi$-action when taking invariants. The positive-degree cohomology of $G$ on $\ell$-primary modules vanishes by averaging. Inflation–restriction identifies the invariant classes after this twist with the rational classes. The local restrictions to the inertia groups and the decomposition group above $\ell$ detect the required vanishing downstairs: their subgroups over $E$ have indices dividing $D$, and restriction is injective in degree one because corestriction multiplies by that index. This proves the descent with the indicated local conditions. Its Tate dual character is $\beta = \kappa_\ell\alpha^{-1} = \psi\kappa_\ell^{j+1}$. In particular $\alpha(\tilde{c}) = -1$, while $\beta(\tilde{c}) = 1$. The Bernoulli index in the bound is $k = j + 1$ and the Bernoulli character is $\chi$, with the same parity as $k$. Strict local family Character twists in the release Eigenspace descent Cyclotomic and Dirichlet character identification

**Strict duality.** Let $S_{\ell,E}$ consist of the places of $E$ over $\ell$ and its archimedean places. Put $A = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-j)$ and $T = \mathbb{Z}_\ell(k)$ with $k = j + 1$. The finite Tate dual of $A[\ell^m]$ is $T/\ell^m T$. Finite-coefficient Poitou–Tate pairs the degree-one localization kernel for $A[\ell^m]$ with the degree-two kernel for $T/\ell^m T$. The archimedean terms in these degrees vanish. Filtered colimits commute with the finite localization diagram, and the finite-coefficient global and local cohomology groups are finite, so the inverse systems satisfy Mittag–Leffler. Taking limits gives the $G$-equivariant identification

$$Ш_\ell(E,j)^\vee \simeq Ш^2_{S_{\ell,E}}(E,T) = \ker\left(H^2(G_{E,S_{\ell,E}},T) \longrightarrow \prod_{w\mid\ell} H^2(E_w,T)\right),$$

where the Pontryagin dual has the contragredient action. The identification is natural under transport of closures and localization data. In particular the character $\psi$ on the strict group corresponds to $\eta = \psi^{-1}$ on its dual, with $\eta(c) = \psi(c) = (-1)^k$ for the minus part. Milne, *Arithmetic Duality Theorems*, I.4.10(a) and I.5.2(b)

Put $U_0 = \mathrm{Spec}\,\mathcal{O}_E[1/\ell]$. The comparison map $H^2(G_{E,S_{\ell,E}},T) \to H^2_{\text{ét}}(U_0,T)$ is injective. For $T/\ell^m T$ this is the degree-two edge injection of the Cartan–Leray spectral sequence over finite étale covers of $U_0$. The direct limit of their degree-one cohomology vanishes, since each class is a torsor for a finite locally constant sheaf and trivializes on a finite étale cover. The étale fundamental group is $G_{E,S_{\ell,E}}$ by the unramified extension correspondence. The finite-coefficient groups in degree one on both sides are finite, so passage to $\ell$-adic coefficients gives the injection by inverse limits. We therefore have

$$\operatorname{length}_O S_\ell(\psi^{-1}\kappa_\ell^{-j}) \le \operatorname{length}_O \big(H^2_{\mathrm{et}}(U_0,T)\otimes_{\mathbb{Z}_\ell} O\big)_{\psi^{-1}}.$$

Here subscripts denote character components. The $\psi$-part of the strict group and the $\psi^{-1}$-part of its dual have the same length once finite. Finite locally constant sheaves and the fundamental group · Geisser–Schmidt, *Poitou–Tate duality for arithmetic schemes*, Theorem B

### 3.4.2. A single integer controls the weighted sum

Put $\Pi_k=\prod_{p\le k+1}p$, with $p$ prime. For a character $\chi$ of $G$ pulled back to the units modulo $N_E$, write $\chi^*$ for its primitive character of conductor $f_\chi\mid N_E$, extended by zero at nonunits. For $k=j+1$ define

$$B_{k,\chi^*}=f_\chi^{k-1}\sum_{u=1}^{f_\chi}\chi^*(u)B_k(u/f_\chi),\qquad A_{N_E,k}(\chi)=N_E\Pi_k B_{k,\chi^*},\qquad b(N_E,k)=16N_E^{k+1}k!.$$

The common character estimate is

$$\operatorname{length}_O S_\ell(\psi^{-1}\kappa_\ell^{-j}) \le v_O(A_{N_E,k}(\chi)). \tag{VI.2}$$

Here $v_O$ counts powers of a uniformizer. In our unramified $O$, $v_O$ agrees with the valuation $v_\ell$ on the chosen embedding into $\overline{\mathbb{Q}}_\ell$, normalized by $v_\ell(\ell)=1$. The release calls the Bernoulli factor $\mathrm{alN}$. The construction of $\mathrm{ThmBcFree}$ takes the finite weighted sum from $\mathrm{PrimeBound}$, which uses $\mathrm{PsiBound}$ at each character. The entry $\mathrm{ML.iv}$ applies $\mathrm{vChar}$ with the reciprocity input. Normalizations Character estimate Length to norm normalization Construction of the two wrappers

**Coates–Sinnott and the torsion correction.** Apply Greither–Popescu, *An Equivariant Main Conjecture in Iwasawa Theory and Applications*, J. Algebraic Geom. **24** (2015), 629–692, Theorem 6.11, to $E/\mathbb{Q}$, $S=\{\infty,2,q,d,\ell\}$, $p=\ell>2$ and $n=k\ge 2$. The set $S$ contains the ramified, infinite and $\ell$-adic places. The field $E(\mu_\ell)$ is abelian over $\mathbb{Q}$ and is already CM, so Ferrero–Washington gives the required $\mu_{E(\mu_\ell)^{\mathrm{CM}},\ell}=0$ for every odd $\ell$. In particular the theorem applies when $\ell\mid N_E$. Write $U=\operatorname{Spec}\mathcal{O}_E[1/(N_E\ell)]$, $W=H^0(E,(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(k))$ and $e_k=(1+(-1)^kc)/2$. Lemma 6.9(1) identifies $W$ with $H^1_{\mathrm{et}}(U,T)_{\mathrm{tors}}$, and $H^2_{\mathrm{et}}(U,T)$ is finite by §6.2 of the same paper. The theorem gives

$$e_k\,\mathrm{Fitt}_{\mathbb{Z}_\ell[G],0}\,H^2_{\mathrm{et}}(U,T)=\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(W)\,\Theta_{S,E/\mathbb{Q}}(1-k).$$

In their convention $\Theta_S(1-k)$ evaluated at $\eta$ is $L_S(1-k,\eta^{-1})$. Thus $\eta=\psi^{-1}$ from duality gives $L_S(1-k,\psi)$. Greither–Popescu, definition of $\Theta_S$ · Lemma 6.9 and Theorem 6.11 · Published paper · Ferrero–Washington, Ann. of Math. **109** (1979), 377–395

The invariant group $W$ is cyclic of finite order at most $\ell^{v_\ell(kN_E\Pi_k)}$. To see this, set $s=v_\ell(N_E)$. Since $E\subseteq\mathbb{Q}(\mu_{N_E})$, the image of $G_E$ on the $\ell$-power roots of unity contains $1+\ell^{\max(1,s)}\mathbb{Z}_\ell$. Any element of this subgroup lifts to a cyclotomic automorphism with other components $1$ that fixes $\mathbb{Q}(\mu_{N_E})$. The element $1+\ell^{\max(1,s)}$ acts on the twist by its $k$-th power. For odd $\ell$ the binomial formula gives $v_\ell((1+\ell^{\max(1,s)})^k-1)=\max(1,s)+v_\ell(k)$. Thus $W$ is contained in the cyclic torsion layer of that exponent, which proves the bound for $s\ge1$. If $s=0$, the same lifting argument gives the full image $\mathbb{Z}_\ell^\times$. If $\ell-1\nmid k$, a lift of a generator modulo $\ell$ has $k$-th power differing from $1$ by a unit, so $W=0$. Otherwise $\ell\le k+1$, and $\Pi_k$ supplies the extra power of $\ell$.

Fitting ideals commute with base change to each factor $O$ of $O[G]$. The annihilator of $W$ does as well, by flat base change to the splitting ring followed by its direct-factor projections. In the $\eta$-component it is $(\ell^{a_{\ell,\eta}})$,

where $a_{\ell,\eta} = \operatorname{length}_O(W \otimes_{\mathbb{Z}_\ell} O)_\eta \le v_\ell(kN_E\Pi_k)$, since each component is cyclic. For a finite module over $O$, its zeroth Fitting ideal is generated by $\ell$ to the power of its length, as follows by decomposing into cyclic modules. For $\eta = \psi^{-1}$ with $\eta(c) = (-1)^k$, we obtain

$$\operatorname{length}_O\big(H^2_{\text{et}}(U,T) \otimes_{\mathbb{Z}_\ell} O\big)_\eta = a_{\ell,\eta} + v_\ell\big(L_S(1-k,\psi)\big).$$

**The removed Euler factors.** For $f = f_\chi$, the function $L_S(1-k,\psi)$ is the primitive $L(1-k,\chi^*)$ multiplied by $1-\chi^*(r)r^{k-1}$ at every finite $r \in S$ with $r \nmid f$. Each such factor is nonzero since $|\chi^*(r)r^{k-1}| > 1$. At $r \mid f$ the primitive Euler factor is already $1$. At $r = \ell$ the correcting factor, if present, is a unit since $k \ge 2$. For the other removed primes, étale localization gives an exact sequence of $G$-modules

$$0 \longrightarrow H^2_{\text{et}}(U_0,T) \longrightarrow H^2_{\text{et}}(U,T) \longrightarrow \bigoplus_{\substack{r \mid N_E \\ r \neq \ell}} \bigoplus_{w \mid r} H^1(\kappa(w), \mathbb{Z}_\ell(k-1)) \longrightarrow 0.$$

Indeed, $U_0$ is regular, the complement of $U$ is a disjoint union of regular closed points of codimension one, and $\ell$ is invertible on $U_0$. Gabber's absolute purity theorem identifies the supported term at $w$ with residue-field cohomology shifted by two degrees and twisted down by one, first for $T/\ell^m T$. All the finite-coefficient groups in this localization sequence are finite, so inverse limits are exact here and compute continuous $\ell$-adic cohomology. At the left, $H^0(\kappa(w), \mathbb{Z}_\ell(k-1)) = 0$ since arithmetic Frobenius acts by $|\kappa(w)|^{k-1} \neq 1$. At the right, $H^3_{\text{et}}(U_0,T) = 0$ by the relative-dimension-zero case of Geisser–Schmidt (0.4) for $U_0$ over itself, with the set of places $S_{\ell,E}$ and each coefficient $T/\ell^m T$. All infinite places are complex and their degree-three groups vanish. Finiteness of the degree-two groups removes the $\varprojlim^1$ term when passing to $T$. Fujiwara, *A Proof of the Absolute Purity Conjecture (after Gabber)*, Adv. Stud. Pure Math. **36** (2002), 153–183 · Geisser–Schmidt, Theorem B and (0.4)

For $w \mid r$, write $D_w, I_w \subset G$ for the decomposition and inertia groups, $F_r \in D_w/I_w$ for arithmetic Frobenius and $f_w = [\kappa(w) : \mathbb{F}_r]$. Procyclic cohomology identifies the residue term at $w$ with $\mathbb{Z}_\ell/(r^{f_w(k-1)} - 1)$. Inertia acts trivially and $F_r$ acts on this quotient by $r^{k-1}$, from its action on the twist and the trivial conjugation action on the absolute Galois group of the residue field. The sum over $w \mid r$ is induced from $D_w$. The $\eta$-component vanishes if $\eta|_{I_w} \neq 1$, which is precisely $r \mid f$. Otherwise projection imposes $r^{k-1} = \eta(F_r)$, giving $O/(r^{k-1} - \eta(F_r))$ since $\eta(F_r)^{f_w} = 1$. Its length is $v_\ell(1-\chi^*(r)r^{k-1})$, as $\chi^*(r) = \eta(F_r)^{-1}$. This accounts for the factors at $2, q, d$ even when the ambient field ramifies and the character does not. If $\ell = q$ or $d$, that prime is already inverted in $U_0$ and its Euler factor was dealt with by the unit calculation at $\ell$.

Subtracting the residue lengths in the exact sequence gives

$$\operatorname{length}_O\big(H^2_{\text{et}}(U_0,T) \otimes_{\mathbb{Z}_\ell} O\big)_{\psi^{-1}} = a_{\ell,\psi^{-1}} + v_\ell(L(1-k,\chi^*)).$$

The primitive value is $L(1-k,\chi^*) = -B_{k,\chi^*}/k$ by the Hurwitz-zeta formula, including the zeta case $f = 1$. In the matching parity $\chi^*(-1) = (-1)^k$, the functional equation relates it by a nonzero factor to $L(k,\overline{\chi^*})$, which is nonzero by the Euler product. The torsion bound pays for the division by $k$, giving $a_{\ell,\psi^{-1}} + v_\ell(L(1-k,\chi^*)) \le v_\ell(N_E\Pi_k B_{k,\chi^*})$. This proves (VI.2) by §3.4.1. The primitive $L$-value can have negative valuation, with the torsion correction supplying the integral Fitting generator. Hurwitz expansion and Dirichlet functional equation · Hurwitz special value

**Proved here using classical Bernoulli identities.** Each $A_{N_E,k}(\chi)$ is integral. Expand $f_\chi^{k-1}B_k(u/f_\chi)$ as $\sum_{i=0}^{k}\binom{k}{i}B_i f_\chi^{i-1}u^{k-i}$. The factor $N_E$ clears the $i=0$ denominator, and $\Pi_k$ clears all denominators of $B_i$ by the von Staudt–Clausen theorem, $B_1=-1/2$, and vanishing of the odd $B_i$ for $i>1$. The values of the character are integral as roots of unity or zero. The preceding nonvanishing argument applies under every embedding, since it preserves the parity. Von Staudt–Clausen Release denominator calculation Release nonvanishing

The classical Fourier series of $B_k(u)$ for $k\ge 2$ gives $\sup_{0\le u\le 1}|B_k(u)|\le 2k!\zeta(k)/(2\pi)^k\le 4k!/(2\pi)^k$. Therefore $|B_{k,\chi^*}|\le 4f_\chi^k k!/(2\pi)^k$. The elementary inequality $\prod_{p\le m}p\le 4^m$ follows by induction. For odd $m=2u+1\ge 3$, the primes in $(u+1,2u+1]$ divide $\binom{2u+1}{u}\le 2^{2u}$, the latter bound using the two equal middle coefficients. At even $m\ge 4$ the product is the previous odd product. Thus $N_E\Pi_k|B_{k,\chi^*}|\le b(N_E,k)$, using $4/(2\pi)<1$. The same bound holds under every complex embedding, since it depends only on $f_\chi\le N_E$ and $k$. In particular the absolute norm from $\mathbb{Q}(\chi^*)$ is a positive integer bounded by $b(N_E,k)^{[\mathbb{Q}(\chi^*):\mathbb{Q}]}$, uniformly in $\ell$. Fourier series of Bernoulli polynomials Archimedean bound in the release

Take the product $M_j=\prod_{\chi(c)=(-1)^{j+1}}A_{N_E,j+1}(\chi)$, where $\chi(c)$ denotes the value of the corresponding character of $G$. Construct it in a cyclotomic field containing all character values. Galois automorphisms permute these characters because they preserve the rational parity sign, and preserve the primitive conductor and Bernoulli formula under conjugation. Thus $M_j$ is a rational algebraic integer, hence a nonzero integer. The product computed in any $\overline{\mathbb{Q}}_\ell$ is its image there. Add (VI.2) over the character summands to obtain

$$\lambda_\ell^-(E,j)\le v_\ell(M_j),\qquad \sum_{\ell\in P}\log\ell\,\lambda_\ell^-(E,j)\le\log|M_j|\le\tfrac{D}{2}\log b(N_E,j+1). \tag{VI.3}$$

Here integrality gives nonnegative valuations for all factors, and the first inequality establishes finiteness before any weighting. Unique factorization gives the middle inequality and shows that only primes dividing $M_j$ can contribute. One may take $P$ to contain all odd prime divisors of $\prod_{j=1}^{t-1}M_j$. Rational integer product Valuation comparison and prime sum Release prime bound

For $2\le k\le t$, use $16\le tN_E$ and $k!\le t^{k+1}$ to bound $\log b(N_E,k)$ by $(k+2)\log(tN_E)$. Interchanging the finite sums in (VI.3) gives the unconditional bound

$$\sum_{\ell\in P}\log\ell\sum_{1\le j<t}\lambda_\ell^-(E,j)\le\frac{D}{4}(t-1)(t+6)\log(tN_E)\le Dt^2\log(tN_E),$$

as asserted in (VI.1). Every odd $\ell$ is prime to $D$ here. For a general target the release also bounds the odd primes dividing $D$ by its separate $\mathrm{StickBound}$. Exact inequalities in the release Complementary degree-dividing terms

### 3.4.3. From reciprocity to the character bound

The released derivation of (VI.2) uses the Gauss-sum and Kolyvagin derivative system stated below, together with the reciprocity and length-descent steps. The dependence on this system is specific to that derivation.

For the strict group in (VI.2) let $f=f_\chi$ and use the primitive character $\chi^*$. Inflating it gives the same $\psi$ on $G_\mathbb{Q}$. In particular $\alpha=\psi^{-1}\kappa_\ell^{-j}$ and all its local conditions are unchanged by this passage. Since $\psi$ has

order prime to $\ell$, its reduction is injective on its values. At a prime $p \mid f$ with $p \neq \ell$, some inertia element has $\alpha(g) - 1$ a unit, by primitivity at $p$. Outside $f\ell$, $\alpha$ is unramified. The release applies the character argument at modulus $f$ and then uses $f \mid N_E$ to enlarge the Bernoulli factor from $f\Pi_k$ to $N_E\Pi_k$. Primitive inertia and the character estimate Passage to the ambient conductor

**Proved here using Kummer theory (residually cyclotomic case).** Suppose $\bar{\alpha} = \bar{\kappa}_\ell$, where bars denote reduction modulo the uniformizer of $O$. Then $\psi$ is unramified outside $\ell$, by injectivity on its values and the equality $\bar{\psi}^{-1}\bar{\kappa}_\ell^{-j} = \bar{\kappa}_\ell$. Put $W = K_O/O$ with action $\alpha$ and let $\pi$ be a uniformizer. The sequence $0 \to W[\pi] \to W \xrightarrow{\pi} W \to 0$ identifies the strict $\pi$-torsion classes with a subgroup of the residual strict group. In detail, globally and on the decomposition group at $\ell$, the invariants in $W$ vanish because some inertia element at $\ell$ has $\kappa_\ell(g) \equiv -1 \bmod \ell$, and then $\alpha(g) - 1$ is a unit. On inertia away from $\ell$, $W$ is trivial and multiplication by $\pi$ on invariants is surjective. The long exact sequence therefore gives the required injectivity on these local groups in degree one. The residual group for $\mathbb{F}_\ell(1)$ is zero: by Kummer theory, a rational class unramified away from $\ell$ has valuations divisible by $\ell$ away from $\ell$, hence is represented by $\ell^a$ modulo $\ell$-th powers, since $\ell$ is odd. Local triviality at $\ell$ forces $\ell \mid a$ as well. Extending coefficients to the residue field of $O$ preserves this argument by linearity. Each $W$-valued cocycle has finite image by continuity and compactness, so its class is killed by a power of $\pi$. Vanishing of the strict $\pi$-torsion forces the entire strict group to vanish. This gives (VI.2) in the branch called exc. Classical Kummer theory Residual case in the main proof Residual vanishing

**Proved here (the Bernoulli valuation in the other branch).** If $\bar{\alpha} \neq \bar{\kappa}_\ell$, choose $g$ where $\alpha(g) - \kappa_\ell(g)$ is a unit. For an integer $m \geq \max(1, v_\ell(f))$, put $f_0 = f/\ell^{v_\ell(f)}$, $L = f_0\ell^m$, and choose $b \in [1, L-1]$ representing the action of $g$ on $\mu_L$. Then $\chi^*(b)b^k - 1$ is a unit. Indeed modulo $\pi$, $\alpha(g) - \kappa_\ell(g)$ is $(\chi^*(b)b^j)^{-1} - b$. For each unit $u \bmod L$ use its representative in $[1, L-1]$, write $y \equiv bu \bmod L$ with $1 \leq y < L$, and put $h_u = (bu - y)/L$. The integer

$$\theta_b(u) = \left\lfloor \frac{bu}{L} \right\rfloor - \left\lfloor \frac{b(L-u)}{L} \right\rfloor = 2h_u - (b-1)$$

is the difference of the two Stickelberger floor coefficients. Choose cyclotomic lifts $\sigma_u$ acting by $u$ on $\mu_L$, and define $z_b = \sum_u \theta_b(u)\alpha(\sigma_u)^{-1}$. Thus $z_b \equiv S_b = \sum_u \theta_b(u)u^j\chi^*(u) \pmod{\ell^m O}$. On the subgroup fixing $\mu_L$ both $\alpha, \beta$ are $1 \bmod \ell^m O$. Use $\delta = \tilde{c}$ at all depths. It acts by $-1$ on $\mu_L$, and $\alpha(\delta) - 1 = -2$ is a unit. Floor coefficients Congruences on lifts Choice of the unit and sum

Write $H(u) = L^{k-1}B_k(u/L)/k$ and $d_k = 2k\Pi_k$. Expansion in Bernoulli numbers gives

$$d_k\big(H(u) - u^k/(kL) + u^{k-1}/2\big) \in L\mathbb{Z}$$

because all terms with Bernoulli index $i \geq 2$ contain $L^{i-1}$ and $d_k$ clears the remaining denominators. Substituting $y = bu - Lh_u$ and reducing the binomial expansion of $y^k$ modulo $L^2$ now gives

$$d_k\big((bu)^j\theta_b(u) + 2H(y) - 2b^kH(u)\big) \in L\mathbb{Z}.$$

Multiply by $\chi^*(bu)$ and sum over units. Since $u \mapsto y$ permutes them, if $\chi_L$ denotes the character modulo $L$ induced by $\chi^*$, the result is the congruence

$$d_k\left(b^j\chi^*(b)S_b - 2(\chi^*(b)b^k - 1)\frac{B_{k,\chi_L}}{k}\right) \in \ell^m O. \tag{VI.4}$$

In this formula $B_{k,\chi_L}$ is evaluated at the imprimitive modulus $L$. Removing the Euler factors at primes dividing $L$ but absent from $f$ multiplies $B_{k,\chi^*}$ by $\prod_{p\mid L,\, p\nmid f}(1-\chi^*(p)p^{k-1})$. This follows from the classical Dirichlet Euler product and the formula at $1-k$. Here the only possible extra prime is $\ell$, and its factor is an $\ell$-adic unit because $k\geq 2$. Hence $B_{k,\chi_L}$ and $B_{k,\chi^*}$ have the same valuation. Coefficient expansion in the release Summed congruence Extra Euler factor

Choose $m$ large enough that $|\ell^m|_\ell < |a_0 d_k B_{k,\chi^*}/k|_\ell$, for a specified $0\neq a_0\in O$. The right side is positive by §3.4.2, and $|a_0|_\ell, |d_k|_\ell \leq 1$. Equation (VI.4), the unit $\chi^*(b)b^k-1$, and the oddness of $\ell$ give $|S_b|_\ell = |B_{k,\chi^*}/k|_\ell$ by the strong triangle inequality after dividing the error by $d_k$. The same equality holds for $z_b$ by its congruence with $S_b$. In particular $a_0 z_b \mid \ell^m$ and $z_b \mid f\Pi_k B_{k,\chi^*}$ in $O$, since $f\Pi_k k$ is integral. This is the numerical input to the cocycle bound. There is no factor polynomial in the auxiliary modulus $L$ in the final valuation. Choice of depth and divisibility outputs Norm comparison and final divisibilities

**Classical reciprocity input and release assertions.** The remaining step is a bound on lengths of normalized cocycles, using an auxiliary family of Gauss sums and Kolyvagin derivatives. The release obtains its reciprocity statement $\mathrm{Rec}$ from Poitou–Tate reciprocity and local unramified orthogonality in the Tate pairing at primes away from $\ell$. Its needed consequence pairs an $\alpha$-cocycle $z$ with a $\beta$-cocycle $\nu$ killed by a common nonzero $a$. Both are unramified off a finite allowed set. At the allowed primes $z$ is locally a coboundary in $W[a]$ except at one auxiliary prime $r'\neq\ell$, and both characters are trivial modulo $a$ on the decomposition group at $r'$. There $z$ vanishes on inertia. If $\nu(\tau)=yw_0$ for an inertia element $\tau$ at $r'$, with $w_0$ of annihilator $(a)$, then $yz(\phi)=0$ for Frobenius $\phi$. The cup product uses $\alpha\beta=\kappa_\ell$. Its local invariants at the unramified places vanish by orthogonality, and the places where $z$ is a coboundary contribute zero. Global reciprocity leaves the unramified–inertia pairing at $r'$. Poitou–Tate and local duality, Milne, *Arithmetic Duality Theorems*, Chapter I Unramified orthogonality Reciprocity interface Reduction to finite coefficients and duality Input wiring

**Proved here using classical duality (the finite-coefficient step).** Identify $W[a]$ with $B=O/(a)$ by choosing $w_0$. The case of a unit $a$ is trivial. Otherwise, because $O$ is unramified over $\mathbb{Z}_\ell$, write $a=u\ell^h$ with $u\in O^\times$ and $h=v_O(a)\geq 1$, and identify $B$ with $O/(\ell^h)$. Thus $\ell^h B=0$, and this exponent is exact. Write $z,\nu$ in $B$-coordinates as $z_B,\nu_B$, so that their product cup class has coefficients in $B(1)$. For each $\mathbb{Z}/\ell^h$-linear functional $\lambda: B\to\mathbb{Z}/\ell^h$, push the cup class to $\mu_{\ell^h}$ by applying $\lambda$ to the underlying coefficient and retaining its Tate twist. The global sum of the local invariants is zero. Outside the allowed finite set both cocycles are unramified, so the cup class factors through the procyclic unramified quotient, whose $\ell$-cohomological dimension is one. At the other allowed primes the class vanishes because $z_B$ is a coboundary there in $B$. At the infinite place the positive-degree cohomology is zero for coefficients of odd order. Thus the pushed local cup class at $r'$ has zero invariant and is zero by local Tate duality.

At $r'$ both cocycles are additive because $\alpha$ and $\beta$ are trivial on the decomposition group modulo $a$, so $\kappa_\ell=\alpha\beta$ is also trivial on $B(1)$. For an unramified Frobenius $\phi$, write $z_B(s)=\eta(s)z_B(\phi)$, where $\eta$ is the unramified additive class with $\eta(\phi)=1$ in $\mathbb{Z}/\ell^h$. Consequently the vanishing just proved is the vanishing of the pairing of $\eta$ with $s\mapsto\lambda(\nu_B(s)z_B(\phi))$, interpreted with the cyclotomic action on $\mu_{\ell^h}$. This map is a cocycle there because the cyclotomic action on $B(1)$ is trivial at $r'$. The unramified classes with trivial $\mathbb{Z}/\ell^h$ coefficients are generated by $\eta$. Unramified orthogonality for the dual coefficient forces $\lambda(\nu_B(\tau)z_B(\phi))=0$ on inertia. The functionals separate points of $B$, by the decomposition of finite abelian $\ell$-groups into cyclic groups. Thus $\nu_B(\tau)z_B(\phi)=0$, which is the assertion used in the length

descent. Global cup class and removal of the other places Local argument with the unramified generator Separation by finite-coefficient functionals

**Release input (Gauss-sum and Kolyvagin derivative system).** Fix $\ell, m, L, \alpha, \beta, b$ as above. Take a prime $r \equiv 1 \bmod L$ by classical Dirichlet, choose a place above $r$, and take $w \in K_O/O$ with annihilator $(\ell^m)$. Put $U = \mathrm{Gal}(\Omega/\mathbb{Q}(\mu_L))$. For every list $I$ of distinct admissible primes $r'$, where $r' \equiv 1 \bmod L$, $r' \neq r$ and $\ell \nmid \mathrm{ord}_{r'}(r)$, the system gives a continuous additive map $\nu_I : U \to K_O/O$ killed by $\ell^m$. These maps are $\beta$-equivariant under conjugation and vanish on inertia outside $Lr\prod I$. The initial map takes some inertia element at the chosen place over $r$ to a unit times $z_b w$. For each fresh admissible $r'$, at every place above $r'$ and every Frobenius lift $\phi$ there, some inertia element has new value equal to a unit times $\nu_I(\phi)$. Here adjoining a prime means prepending it to the list. The decomposition groups at these primes lie in $U$ by splitting. Admissibility The derived system and its local identities Existence statement

The release chooses a degree-one prime in an inverse ideal class, uses a Jacobi sum at that prime, and divides out its contribution from the Stickelberger element for a generator of the product ideal. The resulting elements in $\mathbb{Q}(\mu_{L\prod I})$ satisfy norm and reduction relations up to roots of unity. The release corrects the relations by applying $X \mapsto g_0(X)/X^{a_*}$ and taking a $2f_0\prod I$-th power, where $a_* \equiv \kappa_\ell(g_0) \bmod \ell^m$ and $\alpha(g_0) - \kappa_\ell(g_0)$ is a unit. It then takes the iterated Kolyvagin derivatives, extracts the associated Kummer maps, and projects by a sum weighted by $\beta$. The initial value is $z_b w$ times the unit $2f_0(\alpha(g_0) - a_*)^2$. This is the valuation passed to length descent. The Jacobi-sum identities, the correction of the norm and reduction relations, and the Kummer derivative construction with its local identities are the untranslated inputs. Jacobi-sum and ideal-class construction Norm and reduction up to roots of unity Correction and derivatives Successor value Initial unit and projected floor sum

**Proved here using classical Chebotarev (simultaneous auxiliary prime choice).** Let $z_U : U \to K_O/O$ be a continuous nonzero additive map killed by $\ell^m$ and equivariant for $\alpha$. Let $V_I$ be the $O$-span of $\nu_I(U)$, a cyclic submodule of $W[\ell^m]$. Choose $\rho^{\ell^m} = r$ and let $H \subset U$ be the kernel of the homomorphism $\vartheta(u) = u(\rho)/\rho$ into $\mu_{\ell^m}$. For $g \in G_{\mathbb{Q}}$, $\vartheta(gug^{-1}) = g(\vartheta(u))$, since $g^{-1}(\rho)/\rho$ is fixed by $u$. If an additive map $F$ on $U$ is equivariant for $\gamma$, killed by $\ell^m$, and zero on $H$, it factors through the cyclic image of $\vartheta$. Conjugation by $g$ on that image is the $\kappa_\ell(g)$-power map, so $(\gamma(g) - \kappa_\ell(g))F = 0$. Equivariance on the radical quotient Power action computation

Let $B_z = \ker z_U$. The unit difference $\alpha(g_0) - \kappa_\ell(g_0)$ implies $H \not\subset B_z$. If $V_I \neq 0$, let $K$ be the preimage of $\ell V_I$ under $\nu_I$. If $H \subseteq K$, apply the same quotient argument to $\nu_I \bmod \ell V_I$, with $\gamma = \beta$ and $g = \tilde{c}$. The difference $\beta(\tilde{c}) - \kappa_\ell(\tilde{c}) = 2$ is a unit, so all values would lie in $\ell V_I$, contradicting that they span $V_I$. Each value outside $\ell V_I$ generates $V_I$. If $V_I = 0$, use $K = B_z$. Finally let $C = \mathrm{Gal}(\Omega/\mathbb{Q}(\mu_{L\ell}))$. Take $u_1 \in U$ acting by $1 + L$ on $\mu_{L\ell}$. The element $\tilde{c}u_1\tilde{c}^{-1}u_1$ lies in $H$ by inversion of $\vartheta(u_1)$. It acts by $(1+L)^2 \equiv 1 + 2L \not\equiv 1 \bmod L\ell$, since $\ell \mid L$ is odd. Hence $H \not\subset C$. Next cyclotomic layer Nonzero and generating values on the radical stabilizer

Suppose $B_z, K, C$ cover $H$. A group cannot be the union of two proper subgroups, since multiplying elements exclusive to each gives an element outside both. Choose $x \in H$ outside $B_z, C$, and $y \in H$ outside $K, C$. Then $x \in K,\ y \in B_z,\ xy \in C$. By cancellation $x^2y$ belongs to neither $K$ nor $C$, so $x^2y \in B_z$. It follows that $2z_U(x) = 0$, contradicting odd-order coefficients and $x \notin B_z$. Thus the open set $H \setminus (B_z \cup K \cup C)$ is nonempty. Take a coset of an open normal subgroup contained in it and apply Chebotarev to the class of its image in the finite quotient. Choose the place and Frobenius lift $\phi$ realizing that coset, avoiding the primes dividing $Lr\prod I$ and any other given finite set of primes. At the resulting

prime $r'$, fixing $\mu_L$ and moving $\mu_{L\ell}$ gives $r' = 1 + Ls$ with $\ell \nmid s$. The reduction of $\rho$ is a unit in $\mathbb{F}_{r'}$ because $\phi$ fixes $\rho$. Hence $r^{f_0 s} \equiv 1 \bmod r'$ by $\rho^{\ell^m} = r$, and $\ell \nmid \mathrm{ord}_{r'}(r)$. This gives a fresh admissible prime with $z_U(\phi) \neq 0$ and $\nu_I(\phi)$ generating $V_I$. Subgroup avoidance Admissibility Open Frobenius conditions Simultaneous choice

**Proved here (extension of the derived maps).** The derived maps are additive on $U$. To pair them with $z$ globally, choose $\delta'$ with $\beta(\delta') - 1$ a unit, which exists since $\bar{\alpha} \neq \bar{\kappa}_\ell$. Define

$$n_I(g) = (\beta(\delta') - 1)^{-1} \nu_I(\delta' g \delta'^{-1} g^{-1}).$$

The commutator is in $U$ because $G_{\mathbb{Q}}/U$ is abelian. The commutator of $\delta'$ with $gh$ is the product of its commutator with $g$ and the conjugate by $g$ of its commutator with $h$. Additivity and equivariance give the $\beta$-cocycle identity for $n_I$. For $g \in U$, equivariance and additivity give $n_I(g) = \nu_I(g)$. Continuity follows from that of $\nu_I$. This extension takes values in the cyclic $O$-span of $\nu_I(U)$, and it vanishes on inertia outside $Lr \prod I$ because that inertia lies in $U$. Thus the annihilator used in the reciprocity step kills the full cocycle. Global extension and annihilator

**Proved here from these inputs (length descent).** Normalize a strict cocycle by subtracting the coboundary of $(\alpha(\delta) - 1)^{-1} z(\delta)$, so that its new value at $\delta$ is zero. Each normalized class is represented at some finite level $a_0 \neq 0$, since it has finite image. Include a local primitive at $\ell$ among the elements killed by $a_0$. At each $p \mid f$ away from $\ell$, zero restriction to inertia and a unit value of $\alpha - 1$ there force the same local primitive on the whole decomposition group by the cocycle identity and normality of inertia. That primitive is killed by $a_0$, by evaluating at the unit difference. This gives precisely the normalized $a_0$-torsion cocycle conditions of $\mathrm{Sh}\mathrm{T}$ at the primes dividing $L$. These conditions form a directed union as $v_O(a_0)$ increases. A common finite bound on all these modules bounds the Selmer group, since every finite strictly increasing chain in it is witnessed in some finite level. Normalized cocycles and local conditions Directed limit comparison

Fix $a_0$ and choose $m$ as above. Put $J = v_O(\ell^m)$ and $u = \ell^m / z_b \in O$. Let $A_I$ generate the annihilator of $\nu_I(U)$, and let $S_I$ be the normalized module with extra vanishing on decomposition groups at $r$ and the primes of $I$. Thus $v_O(A_I) \leq J$. A normalized cocycle vanishing on $U$ descends to the abelian quotient $G_{\mathbb{Q}}/U$ on $W[a_0]$. Commuting any element with $\delta$ in the cocycle identity gives $(\alpha(\delta) - 1) z(g) = 0$. Hence a nonzero normalized cocycle is nonzero on $U$. On $U$ it is additive and $\alpha$-equivariant because $\alpha$ acts trivially on $W[a_0]$ there. At a prime split modulo $L$ and away from $L$, vanishing on inertia and at Frobenius gives vanishing on the decomposition group. This holds at all primes above it by the cocycle conjugation formula, since $\alpha$ acts trivially on $W[a_0]$ at the split decomposition groups. For a fresh prime from the simultaneous choice, the derived system makes the new inertia value have the same annihilator as $\nu_I(\phi)$. This annihilator is $(A_I)$. Write $A_{I \cup \{r'\}} = A_I B$. The initial inertia value has annihilator $(\ell^m / z_b)$, so $a_0 \mid A_\varnothing$. The successive divisibilities then ensure that the new annihilator kills both cocycles. In a cyclic coefficient submodule with generator $w_0$ of annihilator $(A_I B)$, this inertia value equals a unit times $B w_0$, by the DVR classification. Reciprocity therefore kills the evaluation $z(\phi)$ by $B$. Evaluation has kernel contained in $S_{I \cup \{r'\}}$ and image in $W[B] \simeq O/(B)$. Hence

$$\mathrm{length}_O S_I + v_O(A_I) \leq \mathrm{length}_O S_{I \cup \{r'\}} + v_O(A_{I \cup \{r'\}}).$$

Whenever $S_I \neq 0$, the chosen evaluation is nonzero, so $v_O(B) \geq 1$. The initial inertia value $z_b w$ has annihilator $(u)$, with $a_0 \mid u$. The same reciprocity argument at $r$ bounds $\mathrm{length}_O \mathrm{Sh}\mathrm{T}$ by

$\operatorname{length}_O S_\varnothing + v_O(A_\varnothing) - v_O(u)$. Along the iteration the annihilator exponents grow strictly and are bounded by $J$, so a module $S_I = 0$ is reached. The inequalities telescope to $\operatorname{length}_O \mathrm{Sh}\mathrm{T} \le J - v_O(u) = v_O(z_b)$. Extended lengths pose no problem because the terminal bound is finite. As $v_O(z_b) = v_O(B_{k,\chi^*}/k) \le v_O(A_{N_E,k}(\chi))$, this gives (VI.2) for the second branch. Cocycle restriction and vanishing criterion Reciprocity step One length increment Initial term and length descent

#### 3.4.4. Placement in the ordinary window

In Paper II's notation the cutoff before rescaling is $v$, the small positive parameter is $\varepsilon$, and $t = \lfloor v^{401} \rfloor$. The field above has $D = 2^{e(v)+2} \le 8v^{2+8\varepsilon}$. On the scale in Paper II, $\log(N_E t) \le 2v^{\varepsilon/4}$ eventually, uniformly in the pair. The unconditional minus estimate $Dt^2 \log(tN_E)$ is therefore at most $16v^{804+33\varepsilon/4}$, within the allowance $C' \exp(8v^{\varepsilon/4})$ for sufficiently large $v$ at each fixed $\varepsilon > 0$. The bound holds pointwise on every such host and contributes no exceptional twists to the interval choice. The total weight is $B_2 \log 2 + B_- + B_+$, with $B_2$ the unweighted dyadic budget and $B_+$ the odd plus budget on the same host. The host plus eigenspace has the cohomological sign $(-1)^j$, whereas the idempotent in the Fitting theorem selects $(-1)^{j+1}$. The dyadic and host odd plus bounds are still required under their separate hypotheses. With those bounds supplied, the total is absorbed into $B_* = \exp(v^\varepsilon/2)$, which becomes $\sqrt{b} = \exp(x^\varepsilon/2)$ after rescaling. Paper II, cohomological budgets Assembly of weighted and unweighted parts Final exponential absorption

### 3.5. Which odd plus estimates does ordinary zeta need?

Write $a = e + 2 \ge 4$, $N = 2^a qd$, and $D = [E : \mathbb{Q}] = 2^a$. A character of the host is uniquely $\rho = \eta J_q^u J_d^v$, where $\eta$ is a character of the real dyadic factor and $u, v \in \{0, 1\}$. The three factors are independent by their ramification. Complex conjugation acts on the character by $(-1)^{u+v}$. Thus the plus part in twist $j$ uses $u + v \equiv j \bmod 2$. There are $D/2$ such characters over a splitting coefficient ring. Throughout this section $1 \le j < t$ and $q, d > \max(2t + 1, 7)$. Host character factors and cases

| Count | Coefficient prime, twist and character | Contribution for ordinary zeta |
|---|---|---|
| I | $\ell \notin \{2, 3, 7\}$, $\ell - 1 \nmid 2j$ | Weighted sum over all such $\ell$ and all $D/2$ characters |
| II | $j \ge 2$, $\ell$ odd, $\ell \in \{3, 7\}$ or $\ell - 1 \mid 2j$, and $\rho \ne 1$ | Individual length estimate |
| III | Nontrivial character of odd order | Empty |
| IV | $\ell = 3, 7$, $j \ge 2$ even, $\rho = 1$ | Zero |
| V | $\ell = 3, 7$, $j = 1$, $v_2(\operatorname{cond}\eta) \ge b_\ell$ | Individual length estimate in growing degree |
| Vs | $\ell = 3, 7$, $j = 1$, $v_2(\operatorname{cond}\eta) < b_\ell$ | Fixed-degree estimate proved below |
| Z | $\ell \notin \{2, 3, 7\}$, $\ell - 1 \mid 2j$, $j$ even, $\rho = 1$ | Zero by §3.3a |

Here $b_3 = 14$ and $b_7 = 18$. Indeed the prime-to-$\ell$ part of $\rho$ is all of $\rho$. At every prime in II or IV the $\ell$-component of its conductor is trivial, because $q, d > \max(7, 2t+1)$ and an exterior prime with $\ell - 1 \mid 2j$ satisfies $\ell \le 2j + 1$. Hence the release's even-order test in II selects precisely the nontrivial characters. The reference threshold at $j = 1$ is $R_1 = 16\ell^2 4^{v_2(\ell^2-1)}$. The odd-conductor part of $\rho$ has order at most two and the order of $\ell$ modulo four is two. Substitution gives $9216$ and $200704$, respectively. The number of Vs characters is $2^{\min(a, b_\ell - 1) - 1}$, since there are $2^{m-2}$ real dyadic characters of conductor dividing $2^m$ for $2 \le m \le a$, and two choices of odd quadratic factor at $j = 1$. The outside sum in I includes $\ell = q, d$. Thresholds · Weighted and individual bounds

**The rational strict group.** Let $O$ be a splitting discrete valuation ring of residue characteristic $\ell$, with fraction field $K$, uniformizer $\pi$, residue field $k$, and $e_O = v_O(\ell)$. Set $W = (K/O)(\rho^{-1}\kappa_\ell^{-j})$. Write $S(W)$ for the classes in $H^1(\mathbb{Q}, W)$ trivial on finite inertia away from $\ell$ and on the full decomposition group at $\ell$. The extra decomposition conditions at $p \mid N$, $p \ne \ell$, are automatic. If inertia acts nontrivially, its image has order prime to $\ell$ and $W^{I_p} = 0$. Otherwise Frobenius minus one acts on $W$ by the nonzero scalar $\rho(\mathrm{Frob}_p)^{-1} p^{-j} - 1$. Multiplication by this scalar is surjective. In both cases the unramified local cohomology is zero. Taking the $\rho$-part over $E$ gives $S(W)$ by restriction and corestriction, globally and on the local conditions, since $[E : \mathbb{Q}]$ is invertible in $O$. Its contribution to the sum is $(\log \ell)\, \mathrm{length}_O\, S(W)/e_O$. Rational strict groups and case bounds

For IV it suffices to treat $\rho = 1$. If $\ell - 1 \mid j$, the trivial residual-character proof of §3.3a applies also at $\ell = 3$. Otherwise the residual character $\bar\alpha = \bar\kappa_\ell^{-j}$ is nontrivial on inertia at $\ell$. A class killed by $\pi$ lifts to a cocycle with values in $W[\pi]$. On inertia away from $\ell$ this lift vanishes, since inertia acts trivially on the entire module. At $\ell$ its restriction is a coboundary $\partial b$ in $W$. Evaluating at an element where $\alpha - 1$ is a unit gives $b \in W[\pi]$. Over the field cut out by $\bar\alpha$, the lifted cocycle is therefore an everywhere unramified additive character, trivial above $\ell$. Its field of definition lies in $\mathbb{Q}(\zeta_\ell)$, which has class number one for $\ell = 3, 7$ by §3.3. Extension of ideals followed by norm is multiplication by a divisor of $\ell - 1$, so the residual field also has trivial $\ell$-class group. Inflation–restriction gives zero over $\mathbb{Q}$, and the $\pi$-primary torsion property gives $S(W) = 0$. This covers every even twist used by IV with the same two reference primes.

**The bounded-conductor case at 3 and 7.** Let $F$ be the field fixed by $\ker \rho$. Its degree is the order of $\rho$. In Vs that order is at most $2^{b_\ell - 3}$, since the real dyadic group at conductor $2^{b_\ell - 1}$ is cyclic of that order and the quadratic factor has order two. The degree caps are $2048$ and $32768$, even as $q, d$ grow. Write $\Delta_F$ for the absolute discriminant. Every character of $\mathrm{Gal}(F/\mathbb{Q})$ has conductor dividing $N$, and conductor–discriminant gives $\Delta_F \le N^{[F:\mathbb{Q}]}$. Restriction to $F$ embeds the strict group into the one with $\rho$ trivialized. For $j = 1$ Poitou–Tate duality identifies the dual of the latter strict group, before extending scalars to $O$, with the kernel of the localization map on $H^2(G_{F,S_\ell}, \mathbb{Z}_\ell(2))$. Here $S_\ell$ contains the infinite places and the primes above $\ell$, and modified real cohomology vanishes at odd $\ell$. Apply duality first to finite coefficients and then pass to the direct limit on strict degree-one classes and inverse limit on degree-two classes. The finite-level cohomology over the ring of $S_\ell$-integers is finite, so the inverse systems satisfy the Mittag–Leffler condition. The odd-primary étale $K$-theory comparison for that ring identifies the ambient $H^2$ with the $\ell$-completion of $K_2(\mathcal{O}_F[1/\ell])$. In total degree two the only other possible descent term is $H^0(F, \mathbb{Z}_\ell(1)) = 0$, since a number field contains only finitely many roots of unity. Localization preserves the $\ell$-part of $K_2(\mathcal{O}_F)$ because residue-field $K_2$ vanishes and the residue-field units above $\ell$ have order prime to $\ell$. A summand of order $\ell^s$ acquires $O$-length $se_O$. We have proved

$$\frac{\log \ell}{e_O}\,\mathrm{length}_O\, S(W) \le \log \left|K_2(\mathcal{O}_F)[\ell^\infty]\right|. \tag{VI.5}$$

This argument uses prime-to-$\ell$ restriction and works for ramified coefficient rings. At $\ell = 7$, $F$ is totally imaginary by parity and unramified at 7 by its conductor. Its group of roots of unity has order $w_F$ prime to 7. Emery's bound on $K_2$ away from $6w_F$ gives a bound $O_{[F:\mathbb{Q}]}(\Delta_F^2(\log \Delta_F)^{[F:\mathbb{Q}]-1})$ on the right of (VI.5). For 3 the following integral estimate suffices. Poitou–Tate duality, Ch. I §4 · Tate on $K_2$ of global fields · Weibel, Chapter VI §8, étale comparison for number rings · Emery, Theorem 1.3

**Lemma.** For each fixed $d_0$ there are $A, C > 0$ depending only on $d_0$ such that $\log |K_2(\mathcal{O}_F)| \le A(2\Delta_F)^C$ for $[F:\mathbb{Q}] \le d_0$.

*Proof.* Put $m = [F:\mathbb{Q}]$, $r = 6$, $L = \mathcal{O}_F^r$ and $\Gamma = \mathrm{GL}_r(\mathcal{O}_F)$. Use Ash's contractible well-rounded retract $X$ of positive forms, with their common scale fixed by making the shortest nonzero lattice vectors have length one. Each open cell is specified by the set of shortest vectors in $L$, which spans $F^r$. Stabilizers are finite, and passing to a boundary cell enlarges that set. We give the count that allows small-prime torsion to be retained. All constants until the end of the proof depend only on $m, r$. Well-rounded cells and stable homology, Soulé §§3–4

Choose an $F$-basis $x_1, \dots, x_r$ of shortest vectors and put $L' = \sum_i \mathcal{O}_F x_i$. Packing balls of radius $1/2$ in the real normed lattice shows that its covolume $\mathrm{vol}(L)$ is bounded below. In the chosen basis $\mathrm{vol}(L')$ is a signature constant times $\Delta_F^{r/2}\prod_\sigma \det(H_\sigma)^{1/2}$, with conjugate embeddings counted twice. Here $H_\sigma$ is the local Gram matrix. Its diagonal entries are at most one, so each determinant is at most one. Hence the index $u = [L : L']$ is $O(\Delta_F^{r/2})$. The lower bound on $\mathrm{vol}(L')$ gives $\det(H_\sigma)^{-1} = O(\Delta_F^r)$ at every embedding. For any other shortest vector $\sum b_i x_i$, we have $ub_i \in \mathcal{O}_F$ and $|\sigma b_i| = O(\Delta_F^{r/2})$ by the bound on $H_\sigma^{-1}$. There are polynomially many possible coordinate vectors at each $u$. Indeed an integral element has an embedding of absolute value at least one unless it is zero, by its norm. Packing in the box prescribed for $ub_i$ proves the count.

The coordinate lattice lies between $\mathcal{O}_F^r$ and $u^{-1}\mathcal{O}_F^r$. There are at most $u^{(mr)^2}$ possibilities, since a subgroup of $(\mathbb{Z}/u)^{mr}$ has at most $mr$ generators. In a fixed coordinate lattice a shortest-vector set has at most $3^{mr}$ elements, by disjoint balls of radius $1/2$ around them inside the ball of radius $3/2$. Thus the number of distinct sets in the polynomially bounded pool of coordinates is polynomial in $2\Delta_F$. The lattice and the set determine the original cell up to $\Gamma$. For each face of this cell the original chosen basis stays shortest, so its enlarged set satisfies the same coordinate and packing bounds. The length of a flag of faces is bounded by the dimension of the space of forms plus one. Barycentric subdivision therefore gives polynomially many simplex orbits in every fixed dimension.

Let $Q$ be the image of $\Gamma$ modulo $3\mathcal{O}_F$, so $|Q| \le 3^{mr^2}$. The kernel is torsion-free. Indeed restriction of scalars embeds it in $1 + 3M_{mr}(\mathbb{Z}_3)$, where cubing an element of exact depth $s \ge 1$ raises its depth by exactly one, and a power prime to three preserves depth. Let $EQ$ be the homogeneous bar construction, the contractible free $Q$-complex with ordered $p$-simplices in $Q^{p+1}$. The diagonal action of $\Gamma$ on $X \times EQ$ is free. A stabilizer must lie both in the torsion-free kernel and in a finite stabilizer on $X$. After subdivision the product cells give a free cellular resolution. In total degree three the quotient has $O((2\Delta_F)^C)$ cells, since at most $|Q|^{p+1}$ choices from $EQ$ accompany any orbit from $X$. Each product boundary column has Euclidean norm at most five, by the two simplex boundary formulas, also when faces coincide in the quotient. The torsion of $H_2(\Gamma, \mathbb{Z})$ injects into the torsion of the cokernel of this boundary. Hadamard's

inequality on its nonzero maximal minors bounds the logarithm of its torsion order by the number of columns times $\log 5$.

Finally the Bass stable rank of a ring of integers is at most two. Rank six lies in the integral isomorphism range for homological stability in degree two, so this $H_2$ identifies with stable $H_2$. The elementary stable group is perfect and equals the stable special linear group over a ring of integers. Its universal central extension has kernel $K_2(\mathcal{O}_F)$. In the split determinant sequence for the stable general linear group, units act trivially on elementary-group homology. Indeed on a finite bar cycle conjugation by $g$ agrees with conjugation by $\mathrm{diag}(g, \det(g)^{-1})$ in one larger rank. In the homology spectral sequence, perfectness kills the incoming second differential to bidegree $(0, 2)$ and the determinant section kills the incoming third differential. Consequently $K_2(\mathcal{O}_F)$ embeds integrally into stable $H_2$. This $K_2$ is finite by finite generation and its rational rank. Combining the estimates and taking the maximum over $m \le d_0$ proves the lemma. Stability range, Soulé §4.2 · Stable elementary groups and $K_2$, Weibel, Chapters III–IV

Taking $d_0 = 2048$ at 3 in (VI.5) gives $\frac{\log 3}{e_O}\,\mathrm{length}_O\, S(W) \le (3N)^{C'}$ for a fixed $C'$. There are at most $4096$ such characters at 3 and $65536$ at 7. These factors can be absorbed into one fixed power of $7N$. This proves the full contribution of Vs at conductor one.

**Remaining odd plus estimates.** Count I still requires a bound by $(Nt)^C$ on the *sum* over each finite set of exterior primes at fixed $j$. Counts II and V require individual bounds by $(\ell N j)^C$ with an absolute exponent for this host family. In II the exterior primes are at most $2j + 1$, so summing those bounds over $j < t$, the eligible primes and at most $D/2$ characters costs only a further fixed power of $Nt$. The two reference primes in II and V have the same property. These three estimates would complete the odd plus term of §3.2. Degree grows in V, and both degree and twist may grow in I and II. The exponents of the fixed-degree lemma above depend on degree. Assembly of the exterior sum

## 3.6. Why two Frobenius traces can bound a sum over all exterior primes

Count I asks for an estimate on a sum over an arbitrary finite set of coefficient primes $\ell$, at fixed twist $j$. On the ordinary family the comparison field in this count is the fixed field $K_0 = \mathbb{Q}(\sqrt{-7})$. Its class number is one. An exterior prime in Case I is prime to $7h(K_0)$ and to every host character order. Its local modulus component in a host character is trivial except possibly at $q, d$, where it is quadratic at most. Inertia at 7 acts nontrivially on $K_0$ and trivially on the two cyclotomic fields of conductors $N$ and $\ell$. These facts give the field, order and ramification hypotheses in the release's one-character estimate. Parity and the condition $\ell - 1 \nmid 2j$ are exactly those of §3.5. Hypotheses on the comparison field and character

### The modular form at a split point of Count I

**Proposition.** Let $U$ be the rational rank-two block of a type-II point in Count I with the Galois conditions of the release. Put $M = 7N$, $k = 2j + 1$ and $\theta = \theta_{\mathbb{Q}(\sqrt{-7})}$. For the chosen coefficient embedding there is a primitive cuspidal newform $f$ of weight $k$, nebentype $\theta$ and level $7\,\mathrm{cond}(\rho)^2$ dividing $7N^2$, attached to $U(-j)$ in the inverse arithmetic Frobenius convention. Its polarized rank-two adjoint block has zero rational finite Selmer group. Rank-two block conditions · Local and global level conditions · Pan, Theorem 1.0.2

*Proof.* The representation $U$ is continuous, absolutely irreducible already on the cyclotomic kernel over $K_0$, odd by its determinant $\theta$, and unramified outside $\ell M$. Its residual semisimplification has characters

$\theta\overline{\Psi}, \overline{\Psi}^{-1}$, where $\Psi = \rho\kappa_\ell^j$. These follow from the block conditions, the identity of residual traces and Brauer–Nesbitt (the determinant is fixed and $\ell > 2$). One may work over a finite coefficient extension. For instance the image of the compact global group is covered by its closed subgroups of matrices in the countably many finite extensions of $\mathbb{Q}_\ell$. One such subgroup has interior by Baire, and adjoining the entries of finitely many coset representatives suffices. The same argument applies to the finite list of local characters. Rank-two block conditions

The ordinary flag at $\ell$ has a line and a quotient whose inertial characters, in that order, are $\xi\kappa_\ell^j, \xi\kappa_\ell^{-j}$, with $\xi$ factoring through the action on $\mu_\ell$. Untwist by $\xi$. The two characters are now unramified twists of cyclotomic powers and hence crystalline. Their ratio $D$, line divided by quotient, has Hodge–Tate weight $-2j$ in the convention $\mathrm{HT}(\kappa_\ell) = -1$. Both $H^0(\mathbb{Q}_\ell, D)$ and $H^0(\mathbb{Q}_\ell, D^\vee(1))$ vanish. The local Euler characteristic gives $\dim H^1(\mathbb{Q}_\ell, D) = 1$. The crystalline dimension formula gives $\dim H^1_f(\mathbb{Q}_\ell, D) = 1$, since $\mathrm{Fil}^0 D_{\mathrm{dR}} = 0$. Thus every extension with this flag is crystalline after the finite twist. This proves the potentially semistable hypothesis in Pan's theorem and gives the two distinct weights. Bloch–Kato, §3 · Pan, Theorem 1.0.2

Apply Pan's theorem to $U(-j)^\vee$. Count I has $\ell \geq 5$, and its residual representation remains a sum of two characters. The Hodge–Tate weights of this dual are $\{1-k, 0\}$, so a modular attachment in Deligne's arithmetic Frobenius convention has weight $k$. More explicitly, a one-dimensional geometric twist over $\mathbb{Q}$ is a Tate power times a finite Dirichlet character by class field theory. Absorb the latter into the primitive newform. The gap of the two weights fixes $k$, and their endpoint fixes the Tate normalization. The determinant of the dual is $\theta\kappa_\ell^{2j}$, fixing the nebentype. Dualizing gives exactly the polynomial of $f$ at inverse arithmetic Frobenius on $U(-j)$. Changing the coefficient embedding replaces $f$ by its conjugate. Pan, Theorem 1.0.2 · Newform convention

At $p \neq \ell$ inertia acts diagonally on $U$ through two characters of $\mathrm{Gal}(\mathbb{Q}(\mu_M)/\mathbb{Q})$. Each has local conductor exponent at most $v_p(M)$, also at $2$, by local reciprocity. The monodromy operator is zero and the Artin conductor exponents add. At $p = \ell$ the finite inertial part of the potentially crystalline Weil–Deligne representation is the scalar $\xi$, with monodromy zero. Its conductor exponent is zero if $\ell \nmid M$. Indeed the level condition gives two finite inertial characters of modulus dividing $M$, and their cyclotomic exponents $j, -j$ distinguish them on every open inertia subgroup. Thus $\xi$ agrees on inertia with the corresponding finite character. At $\ell \mid M$ its tame conductor exponent is at most one. The exponent of the two-dimensional representation at every $p$ is consequently at most $2v_p(M)$. Tate twist and duality preserve these conductor exponents. Local–global compatibility for classical newforms identifies them with the newform conductor exponents, giving the asserted divisibility. Local and global level conditions · Carayol on local compatibility away from the coefficient prime · Saito on compatibility at the coefficient prime

For the prescribed inertia of the marked point one can compute the exact conductor. At $p \neq 7, \ell$, the restriction to inertia is conjugate to characters $\rho_p, \rho_p^{-1}$ on the outer block, giving exponent $2a(\rho_p)$, where $a(\rho_p)$ is the conductor exponent. At $7$, inertia over $K_0$ acts trivially and the quotient has order two. Its determinant is the nontrivial quadratic character, so the two eigenvalues are $1, -1$ and the exponent is one. At $\ell$ the scalar $\xi$ on Weil–Deligne inertia equals $\rho|_{I_\ell}$, by the prescribed ordinary characters and their distinct cyclotomic exponents. This again gives exponent $2a(\rho_\ell)$, including $\ell = q, d$. Since $7 \nmid N$, the conductor is $7\,\mathrm{cond}(\rho)^2$. Inertia comparison in the marked frame · Automorphic attachments and the level calculation

The form is non-CM. Otherwise $U$ is induced from a character of a quadratic field $L$, and its trace vanishes on the nonidentity coset of $G_L$. The ratio of the two residual characters is then $-1$ on that entire coset and $1$ on $G_L$, by taking quotients of two elements in the coset. This ratio restricts on $I_\ell$ to $\overline{\kappa}_\ell^{2j}$, since $\theta$ is unramified there and the square of the finite inertia character of $\rho$ is trivial. This restriction is nontrivial by the defining condition of Count I. On the other hand the decomposition group at $\ell$ preserves the ordinary line. In its intersection with $G_L$, an inertia element has two distinct eigenvalues by the infinite cyclotomic image and the two different powers. Its eigenlines must be the two inducing lines. The ordinary line is one of them, so the entire decomposition group fixes it and belongs to $G_L$. This contradicts the nontrivial residual ratio on $I_\ell$. Rank-two block conditions

On the scalar matrices the polarized block has character $\theta$. On the trace-zero matrices its action agrees with $\mathrm{ad}^0\, U$. Over $G_{K_0}$ it is conjugation. At complex conjugation the action is $-JX^tJ$ for the antidiagonal identity $J$, which on trace-zero matrices equals conjugation by $\mathrm{diag}(-1,1) = U(c)$ in the marked basis. Newton–Thorne's Theorem B(1) applies to the non-CM form over $\mathbb{Q}$ and kills the trace-zero adjoint Selmer group. Passing to the dual or a Tate twist preserves this adjoint. A finite Selmer class for the scalar character $\theta$ restricts to an everywhere unramified additive character on $G_{K_0}$. At $\ell$ the crystalline degree-one classes for the trivial coefficient are the unramified ones, by the crystalline dimension formula. The class group of $K_0$ is finite and restriction detects the rational class by corestriction. This kills the scalar contribution. Action on the rank-two block · Newton–Thorne, Theorem B(1)

The forms in this proposition supply the rank-two portion of the finite list in the trace calculation. There are $O(kM^4)$ of them. All embed as normalized forms in $S_k(\Gamma_1(M^2))$, and distinct newforms give linearly independent simultaneous eigensystems away from $M$ by multiplicity one. The index of $\Gamma_1(M^2)$ is at most $M^4$. The valence bound in weight $k$ bounds the dimension by $1 + k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(M^2)]/12$, by bounding the vanishing order at infinity of a nonzero form. The list is stable under conjugation. At either split probe $p = 11, 23$, for $\ell \neq p$, its rank-three trace values are $1 + p^{-j}a_p(f)$. Deligne's bound gives absolute value at most three in every embedding, and multiplication by $p^j$ clears the denominator. If two points give the same form, their restrictions $1 \oplus U|_{G_{K_0}}$ are isomorphic. One may therefore use all these forms, keeping repeated trace values with their system multiplicities, uniformly across coefficient primes. The irreducible rank-three points require their own list. Newform convention · Deligne, Weil I

### The two off-block Selmer groups

**The two vanishings from Kato's Theorem 14.2(1).** At a split type-II point, let $U$ be the rational rank-two block of determinant $\theta$, the quadratic character of $K_0$. The preceding proposition attaches a primitive newform $f$ of weight $k = 2j + 1$ to $U(-j)$. At good primes the attachment compares the Hecke polynomial with **inverse arithmetic Frobenius**. Kato uses this convention in (14.10.3), and (14.10.2) gives determinant $\kappa_\ell^{1-k}\omega_f^{-1}$ for nebentype $\omega_f$. Hence $U \simeq V_\lambda(f)(j)$ after extending the coefficient field at the place $\lambda \mid \ell$ selected by the attachment, by comparison of the unramified polynomials and semisimplicity. Apply Theorem 14.2(1) over the finite abelian extension $K_0/\mathbb{Q}$ with $r = j$. We have $1 \leq j \leq k - 1$ and $2j \neq k$. The finiteness holds at every coefficient place and for every stable lattice in Kato's representation with this twist. Type-II attachment · Attachment convention · Kato, Astérisque 295 (2004), Theorem 14.2(1) · Kato (14.10.2)–(14.10.3)

At every finite $v \nmid \ell$ the local condition is the kernel of restriction to inertia, including at the bad primes. At $v \mid \ell$ it is the kernel of the map to cohomology with coefficients in $B_{\mathrm{cris}} \otimes V_\lambda(f)(j)$, and the degree-one

condition at infinity is zero. For a stable lattice $T$ in $V = V_\lambda(f)(j)$, Kato's torsion Selmer group in §14.1 uses the images of these local subspaces in $H^1(K_{0,v}, V/T)$. Reduction therefore maps $H^1_f(K_0, V)$ into that finite group. Every class in the rational Selmer group is unramified outside a fixed finite set $S$ containing the primes above $\ell$ and the ramification of $V$. In the exact sequence on $G_{K_0,S}$ for $0 \to T \to V \to V/T \to 0$, the kernel of reduction is the image of $H^1(G_{K_0,S}, T)$. Inflation into full global cohomology is injective in degree one for $V/T$. The integral group is finitely generated over the coefficient DVR. Its image cannot have finite index in a positive-dimensional vector space over the fraction field, so finiteness of the reduction image gives $H^1_f(K_0, V) = 0$. Finite coefficient extension preserves these kernels. Kato §14.1, pp. 234–235 · Kato §§14.8–14.9

Restriction and averaging over the quadratic group identify this Selmer group, after coefficient extension, with

$$H^1_f(\mathbb{Q}, U) \oplus H^1_f(\mathbb{Q}, U \otimes \theta).$$

Indeed the invariant and anti-invariant projections correspond to the two summands of $\mathrm{Ind}_{G_{K_0}}^{G_\mathbb{Q}} \mathbf{1}$. The local identifications preserve the finite conditions. Away from $\ell$ restriction to finite-index inertia detects the kernel by restriction and corestriction. At $\ell$ the same restriction test applies after tensoring with $B_{\mathrm{cris}}$. Thus both off-block summands vanish. Equivalently, apply 14.2(1) over $\mathbb{Q}$ to $f$ and the primitive quadratic twist of $f$ at the same twist $j$. Kato §14.6 · The two off-block projections

**The ordinary argument in the release.** For either $V = U$ or $U \otimes \theta$, the Galois hypotheses give odd determinant $\delta = \theta$, absolute irreducibility on $H_0 = G_{K_0(\mu_{\ell^\infty})}$, and a local line $A$ and quotient $B$ with inertia characters $\xi\kappa_\ell^j$ and $\xi\kappa_\ell^{-j}$, where $\xi$ factors through the action on $\mu_\ell$. The automorphic witness is an additional input to the local comparison. It gives a spherical source after twisting by $\phi = \delta^{-1}\xi^{-1}$ and twisting down by $j$. Here $\delta$ is unramified at $\ell$. The normalized character on $A$ evaluates on Frobenius to the unit root, up to inversion, of the good Hecke polynomial of this source. Both roots have complex absolute value $\ell^j$, by purity at good primes. In particular the character of $A$ cannot be $\kappa_\ell$. Equality would force $j = 1$ and trivial finite inertia twist, and would make the normalized Frobenius value a root of unity. The quadratic twist preserves the inertia flag and multiplies the unit root by an unramified sign. Galois hypotheses and witness · Ordinary source comparison · Kato (14.10.5) and Proposition 17.1

For the global class use the newform $h$ attached to $V(-j)$, so $h = f$ or its primitive quadratic twist. Kato's Theorem 12.5(1), at cyclotomic level zero with $r = k - j = j + 1$ and trivial Dirichlet character, identifies the signed period of the local dual exponential with

$$(2\pi i)^{j-1} L^{\{\ell\}}(h^*, j+1)\, \gamma^{(-1)^{j-1}}.$$

Here $h^*$ is the dual newform, the Euler factor at $\ell$ is omitted, and the period vector has nonzero indicated sign component. Kato's Theorem 13.5(1) (Jacquet–Shalika) gives nonvanishing at $j + 1 = (k+1)/2$, the boundary of the absolute-convergence half-plane. The omitted Euler factor causes no zero there, by the root bound (14.10.5). The geometric reciprocity statement for the smoothed constituents after newform projection is Theorem 9.7. By Theorem 12.6 the integral smoothed families span the rational zeta submodule, so at least one fixed family has nonzero dual exponential after specialization. Proposition 8.12 and Example 13.3 give its integral norm relation (13.1.1). The resulting class $y$ is unramified away from $\ell$, and its local projection $y_B \in H^1(\mathbb{Q}_\ell, B)$ is nonzero since the dual exponential on $H^1(A)$ lands in $\mathrm{Fil}^0 D_{\mathrm{dR}}(A) = 0$. For every $n \geq 1$ and every $q \equiv 1 \bmod \ell^n$ outside a fixed finite bad set, corestriction

from conductor $q\ell^a$ to the degree-$\ell^n$ subfield of $\mathbb{Q}(\mu_q)$ gives the tame class with further corestriction $P_q y$. Here $P_q = \det(1 - q^{-1}V(\mathrm{Frob}_q)) \equiv \det(V(\mathrm{Frob}_q) - 1) \bmod \ell^n$, with arithmetic Frobenius. The same family supplies all depths and the classes have the integral unramified condition away from $\ell$, $q$. Kato Theorem 9.7 · Theorems 12.5(1) and 12.6 · (13.1.1) and Example 13.3 · Theorem 13.5(1) · (14.10.5)

Write $V^D = V^*(1) \simeq V \otimes \kappa_\ell \delta^{-1}$ and identify its pairing with $V$ with the wedge pairing in this model. A strict dual class $z$ is unramified away from $\ell$ and zero locally at $\ell$. Scale $y$, $z$ to make their cocycles and the required local primitives integral. The pinned kernel tests give $y|_{N_0}, z|_{N_0} \neq 0$ for a nonzero strict $z$, where $N_0 = \ker V \cap \ker \kappa_\ell$. The first test uses the nonzero quotient class and the unequal $\ell$-inertia weights on the quotient and the unipotent upper entry. For the second, a class vanishing on $N_0$ descends to the quotient where an element of $\ker V$ with nontrivial cyclotomic value is central and acts on $V^D$ by a nonidentity scalar. The cocycle identity then makes it a coboundary. The $H_0$-stable span of the two additive restrictions projects onto both copies of $V$. If the alternating pairing of the two restrictions vanished, that span would be the graph of a scalar by nondegeneracy and absolute irreducibility. On that graph the pairing is twice a nonzero scalar times the wedge. Thus there are $g, h \in N_0$ with $b = \langle y(h), z(g)\rangle - \langle y(g), z(h)\rangle \neq 0$. There is $\gamma \in [H_0, H_0]$ with $D_\gamma = \det(V(\gamma) - 1) \neq 0$. A subgroup of $\mathrm{SL}_2$ consisting entirely of unipotents is either trivial or fixes a unique line. Applied to the normal commutator image, either case would contradict absolute irreducibility. Pinned kernel and commutator tests

At each depth $n$, the cup $y \cup z$ has a primitive $u$ modulo $\ell^n$ on the global group unramified outside a finite set $S$ containing $\ell$ and the representation's ramification. Its local classes vanish by strictness at $\ell$ and the two unramified conditions elsewhere. Poitou–Tate gives the global primitive since the dual obstruction is an everywhere unramified character of $\mathbb{Q}$ with constant coefficients. Put $D_s = \det(V(s) - 1)$ and $H(s) = D_s u(s) - \langle \mathrm{adj}(V(s) - 1)y(s), z(s)\rangle$. The tame norm relation gives $D_s H(s) = 0 \bmod \ell^n$ for $s \in [H_0, H_0]$. In this step Chebotarev approximates $s$ by a Frobenius $\tau$ at a good $q$ whose degree-$\ell^n$ cyclotomic quotient $\psi_q$ at conductor $q$ is trivial at $S$. The splitting requirement comes from adjoining $\ell^n$-th roots of the primes in $S$ and $\mu_{\ell^n}$, an abelian extension after base change to the cyclotomic tower. The first moment $m$ of the tame Shapiro cocycle, corrected by the coboundary in its augmentation, satisfies $dm = -\psi_q \cup P_q y$. The cup cocycle $m \cup z - \psi_q \cup P_q u$ is locally zero away from $q$, hence at $q$ by global reciprocity. Evaluating at a tame inertia element $t$ with $\psi_q(t) = 1$ and its conjugate by $\tau$ gives $(V(\tau) - 1)m(t) = P_q y(\tau)$ and $\langle m(t), z(\tau)\rangle = P_q u(\tau)$. The adjugate and the congruence for $P_q$ give $D_\tau H(\tau) = 0$. Applying the resulting congruence at $\gamma$ and $\gamma[g, h]$, whose matrices agree, gives $D_\gamma^2 b = 0 \bmod \ell^n$ because the cup identity gives $H(\gamma[g, h]) - H(\gamma) = D_\gamma b$. As $n$ is arbitrary, this contradicts $D_\gamma b \neq 0$. Kato §14.9 on duality · Pinned tame and commutator calculation

For an ordinary dual class, its localization at $\ell$ comes from $H^1(\mathbb{Q}_\ell, B^*(1))$. The reciprocity sum for its cup with $y$ gives zero at $\ell$ since the other local terms vanish. At $\ell$ this pairs the lift with $y_B$ in the perfect pairing of the two one-dimensional local groups for $B$ and $B^*(1)$. The lift is zero and the dual class is strict, hence zero. The primal condition at $\ell$ is the image of $H^1(A)$, exactly orthogonal to the dual condition. It has dimension one by local Euler characteristic since $H^0(A) = H^0(B) = H^0(A^*(1)) = 0$. In the Poitou–Tate difference of the ordinary Selmer dimensions, global invariants vanish, unramified local dimensions equal invariant dimensions away from $\ell$, and the contribution $+1$ at $\ell$ cancels $-1$ from the single real invariant. The primal ordinary Selmer group therefore vanishes too. The finite condition lands in the primal

ordinary one since $H^1_f(\mathbb{Q}_\ell, B) = 0$ by the Bloch–Kato dimension formula and $\mathrm{Fil}^0 D_{\mathrm{dR}}(B) = D_{\mathrm{dR}}(B)$. Kato §14.9 and Proposition 14.12 · Pinned ordinary dual argument

### The trace product

Here are the two arithmetic inputs to the deduction. Write $v_\ell(\rho) = \mathrm{length}_O S(W)/e_O$ for the strict length in §3.5 with $\rho$ and $j$ understood. For a good rational prime $p \geq 7$, choose $w \mid p$ in $K_0$, put $Q = \mathrm{N}w$, and let $g$ be inverse arithmetic Frobenius at $w$.

1. For $\ell \neq p$, there is a finite set $X_\rho$ of representatives of distinct admissible Galois representations over $K_0$. Write $\Psi = \rho\kappa_\ell^j$ and $T_\rho = \Psi(g) + 1 + \Psi(g)^{-1}$. If the traces of all admissible representations at $g$ differ from $T_\rho$, the strict length is finite and satisfies

$$v_\ell(\rho) \leq 2 \sum_{\sigma \in X_\rho} \mathrm{ord}_\ell(\mathrm{tr}\,\sigma(g) - T_\rho), \qquad \mathrm{ord}_\ell(\ell) = 1. \tag{VI.6}$$

   This is the cotangent length comparison at the diagonal deformation with characters $(\Psi, 1, \Psi^{-1})$. The two Selmer vanishings proved above remove the off-block tangent components at the modular type-II points. The rank-two adjoint rigidity follows from the preceding proposition. For the irreducible type-I points, the last part of §3.7 deduces rigidity from cuspidal attachment and the deformation conditions. Application of the one-character bound

2. There is a Galois-stable list of complex traces $u_s$ of full Hecke eigensystems at $g$. Its two parts are the cuspidal rank-three systems and the systems with trace $1 + \mathrm{tr}\,U$. For the chosen $K_0$, assume $|S_p| \leq C[N(j+1)]^A$, $|u_s| \leq \sqrt{Q} + 2$ at every embedding, and $Q^j u_s$ algebraically integral. The constants precede the host, twist and test prime. For each coefficient embedding and each $\rho$, the members of $X_\rho$ have their traces at distinct system indices $s$ in this list. Repeated trace **values** are retained with their system multiplicities. The release obtains this input by local–global compatibility and counting systems at controlled levels. Combination of the two sets of systems

**Proposition.** These two inputs imply Count I on the ordinary host, with constants independent of the finite set of coefficient primes.

*Proof.* Take two fixed primes $p = 11, 23$ and enlarge the common threshold so $q, d > 23$. Both split in $K_0$ since their classes $4, 2 \bmod 7$ are squares. In particular $Q = p$. Work first at one of the two primes. Let $\mathcal{X}_j$ be the complex characters of the host of parity $(-1)^j$. Its cardinality is $D/2$ and it is stable under all automorphisms of the algebraic numbers. Set $R = p^j$ and $z_\rho = \rho(p)^{-1}$. For each $\rho$ and each system index define

$$F_{\rho,s} = R(u_s - 1) - z_\rho - R^2 z_\rho^{-1}.$$

This factor is the integral multiple $R$ of the difference between the trace $u_s$ and the diagonal trace $R^{-1}z_\rho + 1 + Rz_\rho^{-1}$. The last summand in $F_{\rho,s}$ has absolute value $R^2$, whereas the first two together have absolute value at most $R(\sqrt{p} + 3) + 1 < R^2$ for $p = 11, 23$ and $R \geq p$. Hence every factor is a nonzero algebraic integer of absolute value at most $2R^2$. Taking their product over both index sets gives a nonzero rational integer $n_p$, since every field automorphism permutes the factors. Thus

$$\log|n_p| \leq (D/2)|S_p|(2j\log p + 1). \tag{VI.7}$$

Fix an exterior $\ell \neq p$ and an isomorphism from the chosen algebraic closure of $\mathbb{Q}_\ell$ to $\mathbb{C}$. It identifies the $\ell$-adic host characters with the complex characters of the same parity after adjoining their values. For a nonsplitting coefficient field, extend it to include the host character values. The normalized length is invariant under this extension. Now each difference in (VI.6) occurs amongst the factors indexed by that character. The factor $R$ is an $\ell$-adic unit. Unused factors have nonnegative valuation, and the indexing keeps every occurrence even if two different characters have identical traces. We obtain

$$\sum_{\rho \in \mathcal{X}_j} v_\ell(\rho) \leq 2 \operatorname{ord}_\ell(n_p). \tag{VI.8}$$

For $\ell = 11$ use $n_{23}$, and for all other $\ell$ use $n_{11}$. The weighted sum over any finite set in Count I is at most $2 \log |n_{11} n_{23}|$ by unique factorization in $\mathbb{Z}$. Equation (VI.7), $D \leq N$ and $j + 1 \leq t$ give the asserted fixed power of $Nt$. In the release the construction is made for the characters of each exact kernel. The complex fibers have sizes $\varphi([G_N : H])$ and are disjoint, so their total on the host is at most $D$. The product proof above uses the union of the parity fibers. Product and subproduct comparison · Two-prime estimate

**An explicit form of the two integers.** Put $m = 2^{a-2}$, and for $p = 11, 23$ set $h_{11} = m$ and $h_{23} = m/2$. The signs $J_q(p), J_d(p)$ disappear from the character products in (VI.7). To see this, take the representative $-p \equiv 1 \bmod 4$ of the class of $p$ in the cyclic real dyadic group. For $b \equiv 1 \bmod 4$, repeated squaring gives $v_2(b^{2^k} - 1) = v_2(b - 1) + k$ if $b \neq 1$ and $b \neq -1$. Here $v_2(-p - 1)$ is two at 11 and three at 23. Thus the dyadic evaluations at $p$ run over the $h_p$-th roots of unity, each $m/h_p$ times. In either parity two pairs of quadratic exponents occur. Each multiplies the dyadic values by a sign, which permutes those roots since $a \geq 4$ makes both orders even. The multiplicities for the full set are therefore two at 11 and four at 23.

For $R = p^j$ define integral polynomials $C_0 = 2, C_1 = X, C_{d+1} = XC_d - R^2 C_{d-1}$. If $X = U + R^2/U$, induction gives $C_d(X) = U^d + (R^2/U)^d$. The monic polynomial $C_h(X) - (1 + R^{2h})$ has the $h$ roots $z + R^2/z$, $z^h = 1$. They are distinct since equality for two unequal roots would imply $zz' = R^2$, whereas $|zz'| = 1$. We obtain the formula

$$n_p = \prod_{s \in S_p} \left[ C_{h_p}\big(R(u_s - 1)\big) - 1 - R^{2h_p} \right]^{2m/h_p}. \tag{VI.9}$$

The eigensystems in this expression still depend on the host level and twist. For any Case I prime other than 11 and 23, (VI.8) can in fact be applied twice. Its length is bounded by twice the valuation of $\gcd(|n_{11}|, |n_{23}|)$. At 11 use the valuation of $n_{23}$ alone, and at 23 that of $n_{11}$ alone. This also identifies a finite set containing every Case I prime of positive length for the specified host and twist. Classification by system indices · Trace estimate for a Galois-stable family of roots

For a split Count I point the rank-two proposition and Kato's theorem give the required adjoint and off-block vanishings and a finite list of traces at the stated level bound. For an irreducible rank-three point, §3.7 gives the adjoint vanishing conditional on cuspidal attachment. Count I still requires the deformation-ring and cotangent comparisons and the realization and count of the rank-three systems at controlled levels.

## 3.7. Trace products for Counts II and V

Use the notation of §3.5 and write $L_{\ell,j,\rho} = (\log \ell) \operatorname{length}_O S(W)/e_O$. The bounds below give the two individual estimates once the source's deformation and automorphy inputs hold on this host. Count statements

**Counting the systems.** For a fixed imaginary quadratic field $F$, the number of cohomological cuspidal systems in rank $r$ at principal level $M$ with coefficient system $V$ is $O_{F,r}(M^{2r^2}\dim V)$. At fixed level $3$ the finitely many arithmetic quotients have finite cell models by Borel–Serre compactification. Passing to level $3M$ lifts each cell at most $(3M)^{2r^2}$ times, by the index bound from reduction into $\mathrm{GL}_r(\mathcal{O}_F/3M)$. The cochain dimension is bounded by the cell count times $\dim V$. Cuspidal cohomology injects into arithmetic cohomology, with each distinct cohomological constituent contributing at least one dimension. Level $M$ invariants are also invariant at level $3M$. In rank four the Weyl formula at each embedding, for $m=(j+1,j,1,0)$, gives

$$\prod_{i<k}\frac{m_i-m_k}{k-i}=\frac{j^2(j^2-1)}{12}.$$

The two embeddings give $O((j+1)^8)$ for the coefficient dimension. The count is therefore $O(M^{32}(j+1)^8)$. Rank three over $\mathbb{Q}(i)$ has count $O(M^{18})$ at $j=1$ because the weight $(1,0,-1)$ is fixed. Level index and cohomology Cuspidal injection and weight criterion Weyl construction Borel–Serre, §§8–11 Franke, Ann. Sci. ENS 31 (1998), 181–279

To count rank-four points over $\mathbb{Q}$, the source fixes 18 imaginary quadratic fields with pairwise coprime discriminants. At most one is ramified at the test prime. At most 16 sign characters can be self-twists of a rank-four irreducible representation, since their intertwining lines in its endomorphisms are independent. Thus some remaining restriction is irreducible. A restriction has at most two lifts up to isomorphism, differing by a sign twist. At the test prime one recovers candidate rational Frobenius eigenvalues by taking roots of order at most two, at bounded cost. The resulting system bound has the same powers of $M, j$. This step uses base change at the stated level, local–global compatibility and Galois conjugation of algebraically normalized Satake data. Equality of full system indices implies isomorphism by Chebotarev and Brauer–Nesbitt. Equal traces at one prime may belong to different systems, and the product retains their multiplicities. CM system classification Rational classification Arthur–Clozel, Ch. 3

**Count II.** Put $c=v_2(\operatorname{cond}\eta)$ and $B=2^{n_{\mathrm{II}}}$. The source chooses an odd primitive character of conductor $B$ and uses the trivial odd-modulus auxiliary character. Its level parameter is $M=\ell NB$, and the three finite characters of the split lift factor through modulus $M$ and have orders dividing $k=2(\ell-1)B$. The odd-conductor part of $\rho$ has order at most two, so $n_0=v_2(\ell^f-1)$ for $f=\operatorname{ord}_{(\mathbb{Z}/4)^\times}(\ell)$. If $\ell\equiv1 \bmod 4$, then $2^{n_0}\le\ell-1$. Otherwise $2^{n_0}\le2(\ell+1)$. Since $2^c\le2^a$ and $\ell^2<qd$ on II, the ceiling in $n_{\mathrm{II}}$ gives

$$B<2\max\{\ell N4^{n_0},64\cdot2^c\ell^3\}\le8\ell N(\ell+1)^2,\qquad M\ll\ell^4N^2,\qquad k\ll\ell^4N.$$

Thus the dyadic cyclotomic field has degree at most $B$. The dyadic roots required in the coefficient ring lie in an unramified extension of $\mathbb{Q}_\ell$. Even the source's combined exponent $n_{\mathrm{II}}+n_{\mathrm{III}}$ costs degree $O(\ell^6N)$, since the unused III parameter satisfies $2^{n_{\mathrm{III}}}=O(\ell^3)$ at conductor one. In any coefficient extension the Selmer length and $e_O$ scale by the same relative ramification index. Dyadic orders and levels Ceilings Construction and level Finite-character exponent Coefficient extension

At $p\ge5$ prime to $M$, the split trace is $T_4=p^{j+1}z_1+p^jz_2+pz_3+1$, with $z_i^k=1$. The admissible rank-four systems have integral trace $s$ with $|s|\le(3\sqrt{p}+1)p^{(j+1)/2}\le(7/2)p^j$. Since $p\ge5$ and $j\ge2$, the leading term of $T_4$ exceeds $|s|+p^j+p+1$ in modulus. Hence $0<|s-T_4|\le p^{2(j+1)}$. Form $A_4$ by multiplying $s-T_4$ over the systems and the conjugates of the phase triple. Galois permutes the integral

factors, so $A_4$ is a nonzero integer. Consecutive use of the cotangent estimate, integrality of the unused factors and $-\log|A_4|_\ell \le \log|A_4|$ gives

$$L_{\ell,j,\rho} \le 2\log|A_4| \ll |\operatorname{Orb}(z_1,z_2,z_3)|\, M^{32}(j+1)^9 \log p. \tag{VI.10}$$

Indeed the cotangent estimate charges twice the sum of the negative $\ell$-adic log norms over distinct irreducible points. Each point uses a distinct system factor and all other factors have $\ell$-adic norm at most one. Split trace Separation and integral product Product and classification Normalized comparison

The source's least good prime $p \ge 5$ is 5 or 7 on this host. The Galois orbit contains at most $\varphi(k) \le k$ phase triples. Equation (VI.10) gives $O(\ell^{132}N^{65}(j+1)^9)$. Alternatively take the least $p \equiv 1 \bmod M$. The phases are all one and Linnik gives $\log p \ll \log(\ell N)$. This gives

$$L_{\ell,j,\rho} \ll \ell^{128}N^{64}(j+1)^9\log(\ell N) \ll (\ell N j)^{67}. \tag{VI.11}$$

For the last bound at an exterior prime, use $\ell \le 2j+1$, $\ell^5 < N^{5/2}$ and $\log(\ell N) \ll \sqrt{N}$ to absorb $\ell^{61}\log(\ell N)$ in $O(j^{56}N^3)$. At the reference primes the first bound is $O(N^{64}(j+1)^9\log N)$. Good-prime conditions Least good prime Linnik's bound, Xylouris

**Count V.** Here $j = 1$ and the dyadic conductor threshold in §3.5 gives the source's rank-three local conditions with level $N$. The sign character cuts out $K = \mathbb{Q}(i)$, of discriminant $-4$. For $p \ge 7$ inert in $K$ and prime to $\ell N$, put $Q = p^2$ and let $g$ be inverse arithmetic Frobenius at the prime of $K$ over $p$. The diagonal characters are $(\rho\kappa_\ell, 1, (\rho\kappa_\ell)^{-1})$, so the trace at $g$ is $T_3 = Q^{-1}z + 1 + Qz^{-1}$, where $z = \rho(p^{-2})$. In the system count $|s| \le \sqrt{Q} + 2$ and $Qs$ is integral. Because $Q > \sqrt{Q} + 3 + Q^{-1}$, every factor $Q(s - T_3)$ is nonzero and has modulus at most $2Q^2$. The product $A_3$ over the Galois-stable rank-three systems and the conjugates of $z$ is a nonzero integer. The broader source norm includes a rank-two branch. For V, the point classifier sends every tested representation into the rank-three branch. The cotangent bound uses twice the valuation sum and the source allows two embeddings of $K$. Within each embedding the classification is injective on the points being counted. Since $Q$ is an $\ell$-adic unit, this gives

$$L_{\ell,1,\rho} \le 4\log|A_3| \ll |\operatorname{Orb}(z)| N^{18}(4\log p + 1). \tag{VI.12}$$

Host data and dyadic condition Trace and classification Rank-three branch Two embeddings Trace bound Integer product

At $\ell = 3$ take $p = 7$. At $\ell = 7$ take 11 or 19 different from $q$. Neither can be $d \equiv 7 \bmod 16$. The odd quadratic factors disappear at $p^{-2}$. In the real dyadic group of order $2^{a-2}$ the square has order dividing $2^{a-3}$, so the phase orbit has size at most $2^{a-4}$. Alternatively choose the least prime in the reduced class $p \equiv -1 \bmod 2^a,\ p \equiv 1 \bmod qd$. It is inert, avoids 3 and 7, and $\log p \ll \log N$ by Linnik. Its phase is one. Thus (VI.12) gives $L_{\ell,1,\rho} \ll N^{18}\min(2^{a-4}, \log N) \ll (\ell N)^{19}$. The coefficient field can split the host characters by an unramified extension of degree dividing $\varphi(2^{a-2})$. Source prime and orbit choices Source phase orbit Linnik's bound

**Inputs to the cotangent comparisons.** In II, the cotangent comparison uses the weight $-1$ cotangent length at the split lift, a finite degree-zero algebra, exclusion of the specified block end, automorphy of all irreducible points, and rigidity modulo symplectic conjugation. The source proves rigidity by passing to an auxiliary CM field and returning an adjoint coboundary to $\mathfrak{sp}_4$. In V the reduced lifting input gives the marked element, finiteness in degree zero and automorphy of the type-I points over $K$ after a Tate twist. The frame comparison uses local tangent conditions, Poitou–Tate, Euler characteristics and vanishing in the

relevant weights. Once the type-I adjoint vanishing below applies, the remaining tangents come from the torus orbit. The frame comparison gives the common bound for the finite-coefficient lengths, which then bounds the divisible strict group. The construction of the reduced lift and the classification at the stated local level remain inputs to (VI.12). II cotangent bound · II rigidity · V reduced lift · V cotangent and finite-coefficient bounds · V assembly

**A common choice of rational primes for all three counts.** For $q, d > 23$ take $p = 11$ unless $\ell = 11$, in which case take $p = 23$. In I these primes split in $\mathbb{Q}(\sqrt{-7})$. In II both avoid every prime in the ramification set dividing $2\ell N$ when used as indicated. In V, where $\ell = 3, 7$, 11 is inert in $\mathbb{Q}(i)$ and its prime has norm $11^2$. The hypotheses on the Galois element used to bound a deformation length in II and V follow from the respective archimedean trace gaps. The uniform field and representation hypotheses in §§3.6–3.7 remain in force.

With this fixed choice, the phase triples of II have at most $k \ll \ell^4 N$ conjugates and $M \ll \ell^4 N^2$. Equation (VI.10) and the bound on the logarithm of each factor give $L_{\ell,j,\rho} \ll \ell^{132} N^{65} (j+1)^9$. In II, $\ell \le \max(7, 2j+1)$. Summing over its primes, over $j < t$ and the host characters gives $O(DN^{65}t^{143})$. For V the phase at 11 has at most $2^{a-4}$ conjugates, so the bound on a single length is $O(N^{19})$ and both reference layers together cost $O(DN^{19})$. Equation (VI.7) summed over $j < t$ gives $O(DN^A t^{A+2})$ for I, with $A$ the exponent in the count of the systems. Together with §3.5 this proves a uniform bound $(Nt)^{C_*}$ for the entire odd plus contribution from the stated deformation and automorphic estimates. The fixed choices display how all three comparisons apply on the same host and to the same positive twist range.

### Adjoint rigidity of the irreducible rank-three points in I and V

Let $\sigma$ be the rank-three representation on $G_K$ at a type-I point, with $K = \mathbb{Q}(\sqrt{-7})$ in I and $K = \mathbb{Q}(i)$ in V. Write $\theta_K$ for the quadratic sign and $A$ for the adjoint of the polarized deformation over $\mathbb{Q}$. The normal form at complex conjugation $c$ has matrix part 1 and sign $-1$, with $\Psi(c) = \rho(-1)(-1)^j = 1$ for the host character $\rho$. If $J$ is the antidiagonal identity matrix, the pairing is $\sigma(g)^t J \sigma(cgc^{-1}) = J$. Thus $A$ is the full endomorphism space, with the conjugation action on $G_K$ and the action $X \mapsto -JX^tJ^{-1}$ at $c$. The attachment required in both counts is $r_{\pi,\iota} \simeq \sigma(-1)$ for a regular algebraic conjugate self-dual cuspidal $\pi$ of $\mathrm{GL}_3(\mathbb{A}_K)$. The Tate twist leaves $A$ unchanged and makes the polarized multiplier $\theta_K \kappa_\ell^{-2}$, which has value $-1$ at $c$. After killing the finite inertia component of $\rho$, the ordinary diagonal has cyclotomic exponents $(j, 0, -j)$. With $\mathrm{HT}(\kappa_\ell) = -1$, the weights of $\sigma(-1)$ are $(1-j, 1, 1+j)$ at each embedding, hence $(0, 1, 2)$ in V. These weights and the symmetric pairing give the normalization of Newton–Thorne's adjoint. I attachment · V attachment · Polarized action · Newton–Thorne §1

**Image at the specified coefficient prime.** In the marked deformation frame $\sigma$ is absolutely irreducible on $G_K$ by type I. The pinned open-subgroup lemmas use the local flag and the diagonal residual trace to upgrade this to absolute irreducibility on every open subgroup of $G_K$. To see the test, let $U$ be the normal core of an open subgroup. A power of a local inertia element with cyclotomic value $1 + \ell$ lies in $U$, kills the finite character and has three distinct eigenvalues. Any $U$-stable proper nonzero subspace is a sum of its eigenlines. If it is a plane, its intersection with a distinct $G_K$-translate gives a $U$-stable line, since a $G_K$-stable plane would contradict type I. The $G_K$-orbit of the line then consists of all three eigenlines, so the global matrices permute them transitively. The local decomposition group preserves the ordinary flag, which is spanned successively by these eigenlines in their flag order. It therefore acts diagonally in this basis. Take a global element $g$ inducing a three-cycle and a local element $x$ with $u = \overline{\Psi}(x)$ and $u^2 \neq 1$. All

three matrices of $gx^n$, for $n = 0, 1, 2$, have trace zero. Reduction in the original integral model with $v = \overline{\Psi}(g)$ would make the three distinct values $vu^n$ roots of $T^2 + T + 1$, a contradiction. I open-subgroup lemma · V open-subgroup lemma

In I the square of the finite inertia component is trivial, including at $\ell = q, d$, and $\ell - 1 \nmid 2j$ supplies $x$ on inertia. In V, both $3$ and $7$ are inert in $K$ and prime to $N$. Take a local arithmetic Frobenius lift of norm $\ell^2$ and adjust it by inertia to have $\kappa_\ell(x) = 1$. Cyclotomic inertia is surjective over the unramified local quadratic extension. The odd quadratic factors of $\rho$ are trivial on $\ell^2$. For dyadic conductor $2^b$ the exact order of the even dyadic factor at $\ell^2$ is $2^{b - v_2(\ell^2 - 1)}$, since the character is faithful on the cyclic group $1 + 4\mathbb{Z}/2^b\mathbb{Z}$. Here $v_2(\ell^2 - 1)$ is $3$ at $3$ and $4$ at $7$, and $b \geq 14, 18$, respectively. Reduction preserves these orders. The root-counting argument thus applies even in characteristic $3$. The same $x$ also witnesses that the upper triangular ratios of the characteristic-zero diagonal characters differ from $\kappa_\ell$ in the V tangent test. Count I conductor and inertia conditions · V local element and inertia subgroup

The identity component of the Zariski closure of $\sigma(G_K)$ acts absolutely irreducibly, since its inverse image is open. Its unipotent radical fixes a nonzero subspace, so it is reductive, and its center acts by scalars. Its connected derived group is therefore irreducible in dimension three. The possibilities for its semisimple Lie algebra inside $\mathfrak{sl}_3$ are $\mathfrak{sl}_3$ and $\mathfrak{sl}_2$ acting by the symmetric square. Two simple factors would require dimension at least four for a faithful irreducible action. Both possibilities have torus elements with three distinct eigenvalues. The Zariski closure of the image of $G_{K(\mu_{\ell^\infty})}$ contains this derived group, since the cyclotomic kernel contains commutators and the commutator map respects Zariski density. The Tate twist is trivial on that kernel. After a finite coefficient enlargement to split the characteristic polynomials, Newton–Thorne **Lemma 2.28(2)** gives an enormous image in the sense of **Definition 2.23**. The argument applies over any finite extension of $K$ by the same open-subgroup test. Newton–Thorne Definition 2.23 and Lemmas 2.22, 2.28

**The finite tangent conditions.** A dual-number deformation $(1 + \epsilon y)R$ of the polarized point $R$ gives a cocycle with values in $A$. Away from $\ell$, the fixed inertia conjugacy class gives an inertia coboundary by differentiation. Where the frame instead kills a finite-index subgroup of inertia, restriction and corestriction in characteristic zero give the same conclusion. At $\ell$ in I the map of the local deformation ring to the dual numbers kills the specified local ideal. The crystalline comparison for that ideal, applied to the dual-number lift, gives crystallinity over the open subgroup killing the sign and the finite local character. In V, the frame gives a triangular dual-number lift over that subgroup with the fixed cyclotomic powers on inertia. The diagonal blocks are crystalline, since their remaining deformation is unramified. The graded Hom modules for extending the flag have positive cyclotomic exponents $j$ or $2j$, with the ratio-$\kappa_\ell$ case excluded by $x$ above. The Hom module in each extension step has $\mathrm{Fil}^0 = 0$ and $H^0 = H^2 = 0$, by the graded characters and local Tate duality. The local Euler characteristic and Bloch–Kato dimension formula identify $H^1_f$ with $H^1$ for these extensions, so the dual-number lift is crystalline over that subgroup. In both cases its self-extension class becomes a coboundary over $B_{\mathrm{cris}}$. Restriction and corestriction for the finite local extension give the Bloch–Kato condition over the original field. These are the unramified and crystalline kernels of Newton–Thorne §1. I local ideal and tangent comparison · I inertia and crystalline kernels · V inertia comparison · V crystalline tangent comparison

**Vanishing given attachment.** Newton–Thorne, *Adjoint Selmer groups of automorphic Galois representations of unitary type*, **Theorem 5.2** (also Theorem A for unitary type), now gives $H^1_f(\mathbb{Q}, A) = 0$. Finite extension of the coefficient field commutes with the local kernels and detects vanishing. If the

polarized attachment is supplied over a finite CM extension $L/K$, the theorem gives vanishing over $L^+$ and restriction injects the original rational Selmer group there by corestriction. Neither step requires a degree prime to $\ell$. The finite tangent cocycle is therefore a global coboundary, corresponding to an infinitesimal change of basis. This gives the type-I rigidity used in (VI.6) and the rigidity test of the V frame used in (VI.12). Newton–Thorne Theorem 5.2 · I cotangent test · V rigidity test

The first construction still needed for this deduction is a polarized cuspidal attachment of $\sigma(-1)$ at the prescribed coefficient place for each type-I point, including $\ell = 3, 7$ in V. The residual semisimplification is a sum of three characters, so restriction or coefficient enlargement leaves it reducible. In particular BLGGT Theorem 4.5.1, which assumes residual irreducibility on the mod-$\ell$ cyclotomic kernel, cannot provide the attachment directly. At $3$ its numerical lower bound $\ell \geq 2(d+1)$ also fails even for residual constituent dimension $d = 1$. The source routes the attachment through an abelian CM field $L$ containing $K$ and then uses solvable automorphic descent. For descent of the attachment, BLGGT Lemma 2.2.2 requires the original polarized representation over $K$ and irreducibility over $L$, supplied here by the open-subgroup test. The trace count further requires the local level witness over $K$. An attachment over an arbitrary finite CM extension suffices for Selmer vanishing, while the pinned descent and level comparison use the linked field and its specified local conditions. BLGGT Theorem 4.5.1 · BLGGT Lemma 2.2.2 · Source construction in I and V · Source descent and local conditions

For the trace count the auxiliary field must also carry a uniform collection of systems. To see the distinction, suppose a single finite Galois extension $L/K$ works for all points under consideration, with the probes split completely and the upper levels controlled. At such a probe the upper and lower traces agree. A representation which stays irreducible on $G_L$ has at most $|\mathrm{Gal}(L/K)|$ possible extensions up to isomorphism once its restriction is fixed. Indeed Schur's lemma makes the ratio of two extensions a character of the quotient. Thus a single Galois-stable list upstairs, counted with this multiplicity, can bound the required product downstairs whenever its size and the field degree meet the budget. Potential automorphy over fields depending on the point gives no such numerical conclusion by itself. For the linked field in the release the residual outer characters are paired by duality, with a marked element on which they differ. The source uses support on the prescribed ordinary Hecke module for the rank-three lifts and a separate symplectic lifting theorem for Count II. Their precise settings and the solvable descent step are set out in the supplement. Higher-rank attachments and descent

### How a rank-three seed reaches the other irreducible points

The comparison for Counts I and V needs every irreducible point of the specified ordinary deformation space on the chosen CM field $L$. Its residual constituents before the common twist are $\bar{\Psi}, 1, \bar{\Psi}^{-1}$. In the rank-two seed the ratio of the two residual characters is $\bar{\Psi}$. Conjugation on traceless two-by-two matrices has one fixed diagonal direction and two off-diagonal directions carrying that ratio and its inverse. This explains the rank-three residual characters of the adjoint seed. Its cuspidal realization supplies one point of a finite Hecke algebra over weight space. To reach other irreducible lifts the construction uses a patched module over deformation rings. Seed and weight computation

One required characteristic-$\ell$ vanishing admits a short proof, also for $\ell = 3$. Let $A$ be the matrix of the marked decomposition-group element on a residual curve over $k'[[t]]$, and $u$ its cyclotomic value. The characteristic polynomial $P$ of $A$ reduces to $(T-c)^n$, with $c \neq 0$ and $\bar{u} \neq 1$. If $uAY = YA$, then $P(uA)Y = YP(A) = 0$. Modulo $t$, $\bar{u}A - cI$ is the sum of the nonzero scalar $(\bar{u}-1)c$ and a nilpotent

matrix. Thus $P(uA)$ is invertible modulo $t$, hence over $k'[[t]]$. This kills $Y$, also over the fraction field. Marked curves and weight ideals

The support argument has a delicate step at a reducible curve. At adjoining marked curves the procedure completes the same finite patched module. At each completion the regular sequence of patch and weight parameters and freeness of the augmentation fibre give dimension $3[L^+ : \mathbb{Q}] + 1 + d_{\mathrm{patch}}$ for each minimal support component. The Galois deformation estimate at a suitable adjoining prime places its lift above a minimal support prime with irreducible generic determinant. Contract this support prime to the common ring and lift it to the next completion. Flatness after localization supplies a lift. Its residue-space fibre of the common module is nonzero because it is an extension of a nonzero vector space. Passing to a minimal support prime there recovers the dimension needed at the next junction. At the last curve the augmentation-fibre identity puts the desired irreducible prime on the original Hecke support. This uses a common patched module throughout the chain. Dimension, contraction and lifting on support

The local charts used in this argument connect adjacent components in dimension at least $3[L^+ : \mathbb{Q}] + 1$. The source proves the corresponding global propagation using the connected core of the framed space and the rank-two support on the reducible locus. Given these chart and tower estimates, an integral irreducible lift then has nonzero fibre in the finite Hecke module by Nakayama. Arithmetic classicality identifies its determinant and ordered inertia characters with a cuspidal specialization, giving the required weights $(1 - j, 1, 1 + j)$. The remaining literature comparison concerns the definite-unitary level tower, its separating classical evaluations and the normalized local Galois parameters at the split places. From local charts to each integral lift · Precise arithmetic inputs

### The rank-four seed in Count II

The four-dimensional seed uses quadratic induction. Let $k_0$ be the real quadratic field in the selected tower where $2$ and $\ell$ split. The input is an ordinary cuspidal Hilbert modular representation over $k_0$ with two-dimensional Galois representation $r_0$. At the two places above $\ell$, its pairs of cyclotomic exponents are $(k + t_0, -t_0)$ and $(k, 0)$, with $k, t_0 \geq 2$ divisible by $\ell - 1$. The reductions of the ordered characters are $(\bar{\xi}, 1)$ at the first place and $(1, \bar{\xi})$ at the second, with $\xi$ the chosen dyadic character. These opposite refinements matter. The weight-one Eisenstein packet in the source chooses both local roots before lifting its Hecke eigenvalues. Achieving the mixed refinement with the specified finite level and ordinary interpolation is an arithmetic input. Hilbert modular seed and the two local refinements

For $s \in G_{\mathbb{Q}} \setminus G_{k_0}$ representing the marked coset, the induced representation restricts to $r_0 \oplus r_0^s$ over $k_0$. In the induced basis $s$ acts by $\left(\begin{smallmatrix} 0 & r_0(s^2) \\ I & 0 \end{smallmatrix}\right)$. Since $\ell$ is odd and the two residual eigenvalues of $r_0(s^2)$ are distinct, the eigenspace projectors are integral in the selected lattice. The change to the four marked lines exhibits the required alternative of residual entries. Before its common twist the four exponents in decreasing order are $k + t_0, k, 0, -t_0$. The two inducing planes remain inequivalent on the relevant CM field. An isomorphism up to a character on the cyclotomic quotient would, by equality of the determinants, have a quadratic twist. Squaring a suitable inertia element with infinite cyclotomic image would then equate the two distinct pairs of powers above. Given the irreducibility of the two restrictions, Clifford theory gives irreducibility of the induction, and quadratic automorphic induction after base change gives a cuspidal rank-four seed. The target point at twist $j$ has exponents $(j + 1, j, 1, 0)$. The passage from the seed to all of those target points uses a projected rank-four Hecke module, the marked residual support

estimate, and classicality at the target weights. Residual entries and symplectic lattice · Conditions on the projected module and descent

## 4. From one host to the points at a cutoff

Fix $\varepsilon, J$ and let $S$ consist of the real cutoffs admitting $\mathrm{WinDatum}$. The scale $s$ records $s.x = x \in S$, $s.J = J$, $s.\varepsilon = \varepsilon$, $s.L = x^{\varepsilon}$ and $s.b = \exp(x^{\varepsilon})$. **Release input (admissible ranks and host).** Given a tuple $(Y, q, d, e)$ at $x$, the rank theorem extends its frame to $R$ with $\mathrm{RankVal}(s, R)$ and the host admissibility condition. In particular $R.t \le \lfloor x^{400} \rfloor$ eventually. The host theorem uses these ranks, the conductor bounds and the splitting conditions of §2. Its output is $H$ with its arithmetic field, two auxiliary primes $g_0, g_1$, two deformation primes, an initial level prime and sets $TT, T_2$ of frame elements. It satisfies the size, local prime class and conductor requirements $\mathrm{HostGood}$. There is a separate simpler conductor-one host theorem with $J = 1$ and a trivial extra requirement. The construction consuming the cohomological windows uses the supplier with the simultaneous conditions below. Rank supply Host data Choice of tuple and common scale Basic host theorem

The extra host requirement $G_\psi$ has two parts. First, $\mathrm{EAtH}(\mathrm{Level.cyc}(\psi), H, TT, x, \varepsilon, 4)$ asserts the kept and exceptional character conditions, accuracy $x^{-1+\varepsilon}|TT|$ in every coset count, and centered-ramp and log-derivative bounds for the bounded characters. Its ideal shares are $4/(3o+1)$ on the identity coset and $3/(3o+1)$ on the others, with $o = |\langle\psi\rangle|$. For $\mathbf{1}$ the single share equals one exactly. The nonempty quadratic twists still need the pair estimates. The release invokes its bounded-twist lemma on the same $q, d$ used by the host. Second, $G_\psi$ asserts $\mathrm{VVle}(H.E, R.t, \sqrt{s.b})$. **Proved here.** The host field satisfies $\mathrm{IsChiField}$ for the chosen $e, q, d$, so this part follows by restricting the pair's budget to $j < R.t$. A point with the same $s, R, H.E, TT$ and arithmetic host carries both assertions. Host analytic interface Proof of both host assertions Exact transport into point data

### The primes and the level ideal on the selected field

Keep the field and the rank data at this cutoff. Write $N = QS\,qd$, $K = \mathbb{Q}(\sqrt{-d})$, $b = \exp(x^{\varepsilon})$. The field construction gives $d \mid N$, $d \equiv 3 \bmod 4$, and $\log N \ll x^{\varepsilon/5}(\log x)^2$. It takes the level ideal in $K$ to be the unit ideal, whose norm is one. Paper II constructs the sets of thick places and tops on this frame. For each shift the thick places keep the same digits, and the weighted exceptional-orbit estimate uses the same tops as the character average. Level ideal and prime set · Thick digits and their translates · Weighted choice of the tops

**Primes represented by a norm in a short interval.** Suppose $d \ge 100$ is prime, $d \equiv 3 \bmod 4$, $M \ge 3$, and put $B = 4dM^2$. There are absolute $c, c' > 0$ such that, whenever $0 < h \le 1$, $\log B \le c\sqrt{\log X}$, and $h \ge e^{-c\sqrt{\log X}}$, at least

$$c' \frac{hX}{B^3 \log X}$$

rational primes $p \in (X, (1+h)X]$ have a generator of norm $p$ congruent to $1 \bmod M\mathcal{O}_K$. Writing that generator as $(a + b_0\sqrt{-d})/2$ gives $a^2 + db_0^2 = 4p$, $M \mid b_0$, and $2M \mid a - b_0 - 2$. Indeed $1, (1+\sqrt{-d})/2$ is an integral basis. These are the congruences required of a norm prime by the release. Norm and congruence conditions · Short-interval input

To prove the estimate, take the ray class field of $K$ of modulus $M\mathcal{O}_K$, and let $h_M$ be its degree over $K$. The ray exact sequence gives $h_M \le h_K M^2$. The imaginary quadratic Minkowski bound gives an integral ideal of norm $O(\sqrt{d})$ in every class. At norm $j$ there are at most the divisor function $\tau(j)$ such ideals. Thus $h_K \ll \sqrt{d}\log(2d)$ and $h_M \ll B$. Every Hecke character in this extension has conductor dividing $M\mathcal{O}_K$, so its analytic conductor parameter in Thorner–Zaman's Theorem 1.4 is bounded by $4dM^2$. The base degree is two. For $X \le u \le 2X$, that theorem counts the prime ideals with trivial Artin symbol by

$$\frac{F(u)}{h_M}\bigl(1 + O(e^{-c_1\sqrt{\log X}})\bigr).$$

Here $F(u) = \mathrm{Li}(u)$ in the absence of an exceptional zero and $F(u) = \mathrm{Li}(u) - \mathrm{Li}(u^\beta)$ otherwise. The displayed error follows from the two terms in the theorem, since $\log X/\log B \ge \sqrt{\log X}/c$. One can take $c_1 > 0$ fixed for $c \le 1$, and then decrease $c$ to meet the theorem's size threshold. Thorner–Zaman, Theorems 1.4 and 3.3

In the exceptional case Stark's bound in their Theorem 3.3 gives $(1-\beta)\log B \gg B^{-2}$. Consequently $1 - u^{\beta-1} \gg B^{-2}$ on this interval. This follows from $1 - e^{-s} \gg \min(1, s)$ for $s \ge 0$ and $\log X \ge \log B$. Integration of $F'(u) = (1-u^{\beta-1})/\log u$ gives $F((1+h)X) - F(X) \gg hXB^{-2}/\log X$. The same bound holds for $F = \mathrm{Li}$. Each endpoint error before division by $h_M$ is $O(Xe^{-c_1\sqrt{\log X}}/\log X)$, since $F(u) \ll X/\log X$. Choose $3c < c_1$. Then $hB^{-2} \ge e^{-3c\sqrt{\log X}}$ absorbs both errors, uniformly once $c$ is small enough. This gives the lower bound with denominator $B^3$ after division by $h_M$. Prime ideals of residue degree two account for $O(\sqrt{X})$ terms, and primes dividing $Md$ for $O(\log B)$. Their total is absorbed because $hB^{-3} \ge e^{-4c\sqrt{\log X}}$. Shrinking $c$ forces the requisite lower bound on $X$ for all permitted $B$. Each remaining rational prime contributes at most two ideals. Trivial ray class means the ideal is generated by an element congruent to one at the modulus. Since the ideal is integral, its generator is integral at every finite place. This proves the estimate. Ray class version in the release · Thorner–Zaman

**The two primes with prescribed types.** Put $Y_0 = (kb)^8$. For $M = N$ the preceding bound with $h = 1$ gives at least $Y_0^{9/10}$ norm primes in $(Y_0, 2Y_0]$ eventually. Indeed $\log(4dN^2) = o(\sqrt{\log Y_0})$ and $3\log(4dN^2) + \log\log Y_0 = o(\log Y_0)$. For a candidate $g$, its excluded coefficient primes form the set

$$\mathcal{B}_r(g) = \{g\} \cup \{\ell \text{ prime}: \ \ell^{4v_\ell(g^a-1)} > g^a \text{ for some } 1 \le a \le r\}.$$

For fixed $a$ at most three primes qualify, since the product of their exact prime powers divides $g^a - 1$. Thus $|\mathcal{B}_r(g)| \le 3r + 1$. Fix $g_0$ and $\ell \in \mathcal{B}_r(g_0)$. For a given $a$, let $w$ be the least positive integer with $\ell^w > Y_0^{a/4}$. Every $g$ for which $a$ witnesses exclusion satisfies $g^a \equiv 1 \bmod \ell^w$. There are at most $2a$ residue classes, by the cyclic unit group at odd primes and the decomposition of dyadic units into a sign and a cyclic group (the moduli 2 and 4 satisfy the same bound). Each class occurs at most $1 + Y_0^{3/4}$ times in this interval. Including $g = \ell$, the number of excluded partners for $g_0$ is at most

$$(3r+1)\bigl(1 + 2r^2(1 + Y_0^{3/4})\bigr) < Y_0^{9/10}$$

eventually, since $r \le x^5$ and $Y_0 \ge e^{8x^\varepsilon}$. A second norm prime $g_1$ therefore has disjoint excluded set. The norm congruence gives $g_i \equiv 1 \bmod N$. For the later level norm prime at size $T_s$, use $M = N(g_0g_1)^2$, $h = 1/V$. Here $1 \le V \le x^{12}$, $\log M = O(x^\varepsilon)$, and $\log T_s \ge c_s m x^\varepsilon$ with fixed $c_s > 0$ and $m \ge x^{1/3}$. Both short-interval hypotheses hold eventually, and the positive lower bound supplies a prime in the required interval. Excluded prime definition and count · Fixed-coefficient count · Two primes and the later level norm prime Level interval

**The deformation primes and the canonical extra prime.** Linnik's theorem gives an absolute $A \geq 1$ such that a prime congruent to $1 \bmod D$ occurs below or at $D^A$ for every $D \geq 2$, absorbing its fixed multiplicative constant into $A$. Apply it successively to $15N$, $15Np_{\mathrm{de}}$, and $Np_{\mathrm{de}}p'_{\mathrm{de}}$. This gives distinct deformation primes $p_{\mathrm{de}}, p'_{\mathrm{de}} \equiv 1 \bmod 15N$ and a provisional level prime in $1 \bmod N$, all bounded by $N^{C_8}$ for an absolute $C_8$. They lie below $g_0, g_1$ eventually. For the final set let $r_0$ be the least prime avoiding $D = N(p_{\mathrm{de}} - 1)(p'_{\mathrm{de}} - 1)$. A prime divisor of $D + 1$ shows $r_0 \leq 2D$. Since $2 \leq N$ and the level ideal has norm one,

$$r_0 N \leq N^{3+2C_8}, \qquad q_0 = \min\{p \text{ prime} : p \equiv 1 \bmod r_0 N\} \leq N^{(3+2C_8)A}.$$

In particular $q_0$ differs from both deformation primes, as $r_0$ divides neither $p_{\mathrm{de}} - 1$ nor $p'_{\mathrm{de}} - 1$. It lies below the type primes for large $x$, because $\log N = o(x^{\varepsilon})$. The five primes split in the abelian host field whose conductor divides $N$, and the resulting level set is $\{g_0, g_1, p_{\mathrm{de}}, p'_{\mathrm{de}}, q_0\}$. This gives the enlarged host constant at every sufficiently large supported cutoff. Least-prime theorem, Xylouris · Host level selection · Canonical prime bound and releveling

## Common weights on the chosen host

**Conditional sampling theorem.** Fix the host at a supported cutoff, including $TT$, $T_2$, and fix the constants in the order required by the uniform base-level projection estimate. At all sufficiently late cutoffs, that estimate implies the existence of a single pair $a, b$ supported on $TT$ satisfying both weight windows and $\mathrm{WOK}$. The cutoff threshold is uniform over eligible hosts and indices. The arithmetic proof uses only the local **mean** loss at large primes. The geometric projection estimate, including its integral rank certificate and depth, is a release input. Local projection and all size and depth checks · Base-level projection statement

Write $a_* = x\exp(x^{\varepsilon})$, $X' = \exp(x^{\varepsilon/3})$, $b_* = a_*/X'$, $\delta = x^{-\varepsilon/8}$, and $Y_0 = x^{13}$ in this sampling argument. Let $N_1 = O_{\varepsilon}(\log x)$ be the bound in Paper II §4 on the number of tops in an orbit window. With the fixed width constant $K.\mathrm{La}$, put $\mathcal{L} = K.\mathrm{La}\, N_1^2 \log a_*$ and $W_b = 2n\delta x^{\varepsilon}\mathcal{L}b_*/k$. The two intervals are

$$I_a = [a_*, a_* + a_*/(kY_0)] \cap \mathbb{Z}, \qquad I_b = [b_*, b_* + W_b] \cap \mathbb{Z}.$$

Both widths exceed one eventually. In the first, $k \leq x^6$. In the second, $2n\delta \geq 1$, $\mathcal{L} \geq 1$ and $b_*/k$ grows faster than any power of $x$. Sample uniformly from the finite nonempty product $I_a^{TT} \times I_b^{TT}$ and extend by zero. These intervals impose $\mathrm{WinOK}$. The constants in the projection estimate are chosen before using the boxes, so the sampling distribution is fixed for all primes on this host. Box and width definitions · Weight windows

At a candidate prime $p < X'$, the cyclotomic coordinate of Frobenius partitions the frame into orbits. The candidates are in the prescribed inert class with orbit order at least $x^{\varepsilon/4}$. In the range $p \geq k^2Y_0^2$ the candidate condition excludes a power of $p$ in a short relative band about $X'$. The signal $S_p$ is $m$ times the sum of the rank losses over the good orbits of Paper II §4. For an orbit with $N_T$ tops, each assignment of the local weights has exactly $(|I_a||I_b|)^{|TT|-N_T}$ extensions. Its marginal is therefore uniform on the prescribed local boxes. The tops in the good orbits are disjoint, with sum of their cardinalities at most $|TT|$. Good orbits · Candidate exclusions and marginals

For $p < k^2 Y_0^2$ the defined loss is bounded by $C N_T k x^{-26} \le N_T x^{-12}$ eventually, with $C$ fixed once the orbit constants are fixed. At large primes the base-level projection estimate gives mean loss at most $C_{\mathrm{pr}} N_T N_1 / Y_0$. Its certificate is uniform in the bounded interior weight coordinates, the local coefficient ring and the target lattice. It holds at the required depth and bounds the least admissible rank loss from above. In transferring the estimate, the orbit test gives $N_T \le 40 n' \delta$ for orbit order $n'$, and the band exclusion and height bounds apply to the prescribed boxes. The supporting proof checks the signed odd-top term in the serving depth and the dimension-weighted change of boxes in the local estimate. Since $C_{\mathrm{pr}} N_1 \le x$ eventually, the large-prime loss has mean at most $N_T x^{-12}$ as well. Local means and quantitative transfer · Serving-depth and numerical bounds

It follows on the single global product that $\mathbb{E} S_p \le m |TT| x^{-12}$ for each candidate and $\Pr(S_p > x^{-\varepsilon}) \le x^{\varepsilon} \mathbb{E} S_p$. Write $u = \log X' = x^{\varepsilon/3}$. Summing over all integers up to $X'$ gives $\sum_p (\log p)/p \le u(2+u)$ for the candidate sum. Thus

$$\begin{aligned}\mathbb{E} \sum_{p \text{ candidate}} \frac{\log p}{p} \left( \mathbf{21}_{\{S_p > x^{-\varepsilon}\}} + 4 S_p \right) &\le 4 (x^{\varepsilon} + 1) m |TT| x^{-12} \, u(2+u) \\ &\le 48 x^{-19/2 + (9 + 2/3 + 1/50)\varepsilon}.\end{aligned}$$

For the last inequality use $m \le x^{1/2 + \varepsilon/100}$, $|TT| \le 2 x^{2 + 8\varepsilon + \varepsilon/100}$ and $u(2+u) \le 3u^2$. The excess saving $\gamma = 1/2 - (9 + 2/3 + 1/50)\varepsilon$ is positive for $\varepsilon < 1/52$. Taking $x^{\gamma} \ge 48$ makes the bound at most $x^{-9}$. A point in the finite product attains at most the mean and supplies $\mathrm{WOK}$. Linearity of expectation allows the restrictions for different primes to overlap arbitrarily. Independence is used only for the coordinates of the weight box. Finite averaging and uniformity

### A determinant estimate on the original interval of weights

The large-prime argument varies one terminal weight while keeping the other weights of its segment fixed. Its Gram matrix $M(z)$ has size $s$, with entries of degree at most $J$ in the integral binomial basis over $\mathbb{Z}_p$. Take $W$ consecutive integers with $1 \le W \le p^h$ and $J < p^h$. Assume $\det M$ is a nonzero polynomial with leading coefficient $l$ and degree $d$, and $v_p(l) + d/(p-1) \le sL$. For a positive integer $E$, let $\beta(z)$ count the Smith factors divisible by $p^E$, including zero factors. Then

$$\frac{1}{W} \sum_{z} \beta(z) \le s \left( \frac{J}{W} + \frac{L + Jh/W}{E} \right).$$

This inequality also counts a weight at which the specialized Gram matrix is singular. Terminal Gram window and application

For the proof, expand $M(z + p^h U)$ in $U$. Its coefficients are integral by the binomial addition formula and $v_p(i) < h$ for $1 \le i \le J$. In Smith coordinates at $z$, a determinant coefficient of degree $i$ uses a nonconstant term from at most $i$ rows. Its valuation is at least $E \max(0, \beta(z) - i)$. Factor $\det M(Z) = l \prod_{\zeta} (Z - \zeta)$ over a finite splitting field, counting roots with multiplicity. Let $d_z$ be the number of roots in $v_p(z - \zeta) \ge h$. Divide each factor of $\det M(z + p^h U)$ by a coefficient of least valuation. After reduction the product has degree exactly $d_z$, since each close root contributes degree one and each distant root degree zero. The coefficient of $U^{d_z}$ therefore has valuation $v_p(l) + \sum_{\zeta} \min(h, v_p(z - \zeta))$. Each root belongs to at most one such close disc. For $0 < u \le h$ the

number of weights in $v_p(z-\zeta) \geq u$ is at most $W/p^{\lceil u \rceil} + 1$. Integrating in $u$, summing over roots, and using $d \leq sJ$ proves the bound. Integral change of variable and Smith estimate

To apply the bound on the original product of integer intervals, the source compares dimensions within each digit class, constructs paired integral columns on each edge, and compares the harmonic sublattices at the prescribed depth. The detailed calculation uses these three inputs to bound the mean rank loss by $C_{\mathrm{pr}} N_T N_1 / Y_0$, where $N_T$ counts the tops in the orbit. The interval enlargement used in the harmonic decomposition costs a factor at most two when transferred back to the original uniform measure. This supplies the local estimate used in the common-weight argument above, conditional on the three integral estimates. Rank and integral comparisons · Recovery of the original mean

## Realization with prescribed weights and level

**Word supply.** Write $k, m, f$ for the corresponding sizes in $R$, and put $\Gamma = \exp(C_\Gamma m^5 f x^\varepsilon)$ and $D_{\mathrm{word}} = \lceil 3\Gamma^2 x^{C_F} \rceil$. The rank estimates give $m^5 f x^\varepsilon \leq 128 x^{9/2+9\varepsilon}$ and $k \geq x^{9/2+10\varepsilon}$. For fixed $C_\Gamma, C_F \geq 0$, $\log D_{\mathrm{word}} = O(x^{9/2+9\varepsilon} + \log x) = o(k)$. Thus there is an injection $\mathrm{Fin}(D_{\mathrm{word}}) \to \{0,1\}^{k-2}$ eventually. Choose it once for the kit. The point provenance gives a bijection from its own words to $\mathrm{Fin}(D_{\mathrm{word}})$, with bits equal to those of the common code followed by zeros. Hence identical word indices yield identical bits across points. The additional word tests $\mathrm{ChCode_at}$ come from the object and numerical hypotheses on the same kit. Rank estimates and injection Word bijections and agreement Further word tests

For the geometric construction fix the weights in $\mathrm{WinOK}$, the word code, and a shift $u_0 < n$. Use $F = E^+$, $f = 2n$, $r = k + m - 1$, $\Delta = \lceil x^{c_D} \rceil$, and $h = \lceil x^{c_h} \rceil$, with $c_D, c_h$ fixed and $c_h \geq 402$. In the ordinary field, finite inertia at $q, d$ can act only on the respective quadratic factor, and inertia at $2$ can act only on the real dyadic factor. No finite inertia contains complex conjugation, which changes both imaginary radicals. Hence $E/F$ is unramified at finite places. Arithmetic and realization theorem

At each real place take $v_i = 4\Delta(k-2-i)$, $0 \leq i \leq k-2$, and $u_{i+1} = v_i - \frac{1}{2} - \Delta c_i$ for the internal coordinates of the word, with $c_i \in \{0,1\}$. The two endpoints at the top $t_0$ are $a(t_0) + (k-1)/2$ and $-b(t_0) - (k-1)/2$. Away from the tops they are $4\Delta(2k-2) + 1/2$ and $-1/2$. For $0 \leq j < m$ set $b_j = B_\tau + 2jh$, $e_j = b_j + h$, where the base $B_\tau$ is $b(t_0) + \lfloor b_*/x \rfloor$, $\lceil \exp(x^\varepsilon) \rceil$, or $\lceil s.w''(n) \rceil$ at a top, thick or thin place, respectively, with $s.w''(n) = b_*/(xn)$. A shift of $B_\tau$ at one thick place by less than $|\mu(E)|$ cancels the infinity-type character on CM torsion. Strict interlacing holds since each internal $u_{i+1}$ lies between $v_i - 4\Delta$ and $v_i$, and the endpoint inequalities use $4k\Delta \leq a_*$, $\lfloor b_*/x \rfloor \geq k$. The $b, e$ gaps are $h \geq (t+3)/2$ since $t \leq x^{400}$ eventually. The compact rank-$k$ dimension is the Weyl product $\prod_{i<j} (u_i - u_j)/(j-i)$. The row parameter uses $v_i$ and $-b_j$, with alternating signs on the latter entries. Its normalized polynomial is $\prod_{i<j} |\lambda_i - \lambda_j|/(j-i)$, with $\lambda = (v_i, -b_j)$. Weight formulas and torsion shift · Row formal-degree polynomial · Prescribed family of weights

At either completely split type prime $g$, choosing one member of a split pair identifies the unitary group with $\mathrm{GL}$ on that member. The other component is the inverse adjoint. Reduction gives the column upper unitriangular subgroup in rank $k$. In the row group of rank $r$, use the upper unitriangular matrices with the central diagonal block of size $k+1$ fixing its last basis vector. This imposes $k$ extra zero entries above the diagonal in that block. The central diagonal block is multiplicative on upper block triangular matrices and

respects inversion, so these matrices form a subgroup. Counting free entries and dividing $|\mathrm{GL}_s(\mathbb{F}_g)| = g^{s(s-1)/2}\prod_{i=1}^{s}(g^i-1)$ by the subgroup orders gives

$$I_{\mathrm{col}}(g) = \prod_{i=1}^{k}(g^i-1), \qquad I_{\mathrm{row}}(g) = g^k\prod_{i=1}^{r}(g^i-1).$$

Each $g_i$ contributes $f$ such indices. The signed tame list at $g_i$ consists of the exponents $0, \pm 2, \dots, \pm 2m$ of a primitive residue unit. Outside $\mathcal{B}_r(g_i)$ its prime-to-$\ell$ order is at least $(g_i-1)/g_i^{1/4} > 4m$ eventually, so the list stays distinct under the prime-to-$\ell$ projection. The disjoint excluded sets ensure that one of the two choices works at each coefficient prime. Exact subgroup and tame calculation · Local level definitions

**Norm-window input and its consequence.** If $M_s$ is the mass of the identity lattice genus in rank $s$, let $d_w$ be $M_k$ times the column indices and the Weyl dimension product for word $w$. Let $D_*$ be $M_r$ times the row indices and the product of the normalized row factors of the pinned construction. These quantities are fixed before the norm primes are chosen. The normalized mass ratio and Vandermonde comparison give the release bound

$$\frac{\log(D_*/d_w)}{fk} \;\geq\; m(\log g_0 + \log g_1) + \frac{1}{f}\sum_{\tau,j}\log(e_j(\tau)v_E/\pi) - C_1(x^{\varepsilon} + m^2x^{\varepsilon}/k),$$

where $v_E \geq \pi$ is the field factor. This inequality and the identification of $d_w$ with a positive integer are pinned inputs to the prose argument. The proof of the bound in the source telescopes the unimodular mass ratios and compares the interlacing Vandermonde factors. Given the bound, the sum is nonnegative, $\log g_i \geq 8x^{\varepsilon}$, and $k \geq m^2,\ m \geq 2|C_1|$ eventually. Thus the quotient is at least $15mx^{\varepsilon}$. The penalty $2\log\Gamma/(fk) \leq mx^{\varepsilon}$ eventually by $2C_\Gamma m^4 \leq k$. Put $S_w = (D_*/d_w)^{1/(fk)}\Gamma^{-2/(fk)}$. This gives $\log S_w \geq mx^{\varepsilon}$. The norm-prime estimate above applies with $X = S_w$, $M = N(g_0g_1)^2$, $h = 1/(fk)$, since $fk \leq x^{12}$, $m \geq x^{1/3}$ and $\log M = O(x^{\varepsilon})$. It gives

$$S_w \leq T(w) \leq (1+1/(fk))S_w, \qquad \pi_T(w)\overline{\pi_T(w)} = T(w), \qquad \pi_T(w) \equiv 1 \pmod{N(g_0g_1)^2\mathcal{O}_K}.$$

The primes split completely in $E$ and are larger than $X'$ and the five initial level primes eventually. The host level set adds the $T(w)$. The choice is per word, with repetition permitted. Mass normalization, window bounds and their hypotheses · Pinned numerical window proof

Write $N_j = k + j(t+1)$, $R_j = r + jt$ and $q_j = \lfloor (m-j)/2 \rfloor$. Take $E$-spaces with diagonal Gram matrix having the indicated number of plus signs first, and give them the self-dual lattices $\mathcal{O}_E^{\mathrm{rank}}$.

| Space | Rank | Signature at each real place |
|---|---|---|
| $X_j, 0 \leq j \leq m$ | $N_j$ | $(N_j, 0)$ |
| $X'$ | $R_m$ | $(R_m, 0)$ |
| $Y_j, 0 \leq j \leq m$ | $R_j$ | $(R_j - q_j, q_j)$ |
| $V_j, 1 \leq j \leq m$ | $r+t$ | $(r+t, 0)$ |
| $W_j, 1 \leq j \leq m$ | $r$ | $(k+j-1, m-j)$ |

Each form $J$ defines its group by $\bar{A}^{\mathsf{t}}JA = J$. The positive forms give compact real factors. Successive $X_j$ add $t+1$ positive lines. For $j < m$, embed $X_j$ into the first $N_j$ coordinates of $Y_j$. The remaining

$m-1-j$ coordinates admit $\lfloor (m-1-j)/2 \rfloor$ hyperbolic pairs $e_+ + e_-, (e_+ - e_-)/2$. A spare negative coordinate occurs exactly when $j$ is even, and the vector $\pi_T e_-$ there has norm $-T$. Here $T$ is the norm prime of the distinguished word. At the top $Y_m = X'$, and the last line of $X_m$ contains the perpendicular vector $\pi_T e$ of norm $T$. The coordinate lattice is self-dual since $J$ is unimodular. Forms, embeddings and hyperbolic frames · Pinned signed model

At $T$ the auxiliary column lattice $M_b = \mathcal{O}_E^{k-1} \oplus \pi_T \mathcal{O}_E$ satisfies $M_b + \bar{\pi}_T M_b^{\#} = \mathcal{O}_E^k$. The self-dual split move is given on the two factors of the selected coordinate by $(T, T^{-1})$. On writing $O = \mathcal{O}_{F,v}$ at $v \mid T$, $O \oplus O$ and $TO \oplus T^{-1}O$ meet in $TO \oplus O$, of index $T$ on each side. At $a$ such places the index on each side is $T^a$, in particular $T^f$ if every place over $T$ is used. Negative or positive orthogonal lines and their fractional ideals extend the lattice models, with the dual ideal on a unit-norm line equal to the inverse conjugate ideal. For a finite adelic isometry $a_f$ the transported lattice is $E^{\mathrm{rank}} \cap a_f \widehat{M}$. Clear denominators of $a_f$ and $a_f^{-1}$ by a single integer $D$. Chinese remaindering in $D^{-1}M/DM$ realizes the prescribed completions and bounds the index on each side of the intersection by $D^{[E:\mathbb{Q}]\,\mathrm{rank}\,M}$. Left rational isometries transport this intersection and the decorated right stabilizer preserves its reductions. For a sequence of finite adelic moves the product of the successive denominator bounds suffices. The shift $u_0$ resets the distinguished host index, with $TT, T_2$, weights and code carried by provenance. Local moves and integral gluing

## The two automorphic passes and the point ledger

**Release inputs (the automorphic realization).** The first global step after the signed and decorated lattice models is the saturated automorphic seed. It is a row reading at the prescribed real weights and finite types, compatible with the column models and their intertwining maps. In the untwisted pass the active coefficient places, selected by the local required content of the lifts, must be a union of fibres over the fixed row coefficient field $L_0$. A whole fibre is selected whenever one of its places is. The first state has empty localization and zero local depth. Its pinned construction invokes the fine adelic reading and its extension in the ambient coefficient field, the lower-unitary multiplicity equality and the weight-specific rational row test. That test uses the specified local Weil-vector matching, the theta integral comparison and its recurrences for adapted plane changes. The general discrete-series and Siegel–Weil theorems supply parts of the comparison. The relative multiplicity and rationality statements for these particular vectors, levels and normalizations are used as inputs here. Saturated seed and precise spectral hypotheses

In the first pass choose the fixed fields, split-place orientations and true columns. The twist characters initially equal $1$ and the central tuples include the two distinguished characters as well as $1$. Keeping the prescribed Galois level, install the rational row lattice, cusp data, defining forms, induced series and moduli data. The strict-henselian and finite spectral steps give a finite-level state with the same support. The discrete-series replacement carries the actual row evaluation and its norm identity, the real volume normalization, the local types and the column models. Join the row, column, division and theta data using the matched characters and intertwiners. This gives the first core and its export. Construction through the spectral replacements · First and second pass

In the second pass use that export to select finite-order twists at the active places with conjugation, agreement over the fixed coefficient field and the local avoidance conditions. Enlarge the ambient coefficient field for their values while preserving the fixed embedded fields and the rational prime list. Run the row, spectral and kernel constructions again with these twists. The resulting core supplies good local

twists for every required exponent prime to the relevant residue characteristic. The normalized volume and column-count comparisons identify its actual pre-level sizes with $D_*, d_w$. These comparisons, and the column denominator choice $\mathrm{MinkDiv}$, enter the recursive object theorem. Second pass and the required count and volume equalities · Actual two-pass assembly

The resulting $\mathrm{RecOK}$ records $\Delta$, the error parameter $6x^{-C_F}$, finite active support, vanishing lift corrections off that support, fullness of the ramified division module, the column denominator and tame conditions, and local comparison at $T(w)$. For any valid size record at the same scale and weights, $\mathrm{LedOK}$ supplies the local containment and denominator bounds for the canonical parameters. After refilling the parameters, glue the pulled-back kernel lattice at served places, the prescribed comparison lattice above $T(w)$, and the ramified division lattice at all other places. The exceptional set is finite since the served primes are at most $X'$. The local saturation and comparison bounds give the stated localizations and allow the ledger to record their lengths divided by ramification indices. The division-lattice replacement preserves $QD$, which requires used tuples, good twists with conjugation and the specified prime-to-residue-characteristic powers, and stability of the row lattice under the full coefficient integer ring. This gives the object at the original scale and weights, with provenance $\mathrm{Pt.From}$ and a canonical instance. Exact ledger gluing and $QD$ · Extraction of the point

### Saturation of the reference support

At the first pass fix the arithmetic host, the frames, and the prime norms. In the CM field $E$, there are only finitely many integral generators of a prescribed norm $\pi\bar{\pi} = T$. In every complex embedding the generator has modulus $\sqrt{T}$, and the Minkowski lattice meets this compact set in finitely many points. Choose a fine automorphic realization for each required generator tuple, assuming these realizations exist. Their reference fields lie in one finite coefficient field $L_0$. Each pair of full lattices in a common vector space is commensurable, so the selected comparisons have finite prime support. Reference choices and the two index pairs

Let $A \subseteq L_0 \subseteq L$ be the chosen reference field, fixed coefficient field and enlarged coefficient field in their distinguished embeddings. Write $U_A$ for the support of the two signed reference indices, including their conjugates. For either pair $M, N$, scalar extension carries $M \cap N$ to the intersection after extension by flatness. At $w \mid v$ each finite quotient length is multiplied by $e(w/v)$, as one sees on a residue-field composition factor. Thus the signed difference also scales by $e(w/v)$. The Hermitian dual commutes with this extension in a basis by the inverse Gram matrix and its conjugate scalar action. Given the source's simultaneous base change of the two reference pairs, the active set in $L$ is the inverse image of $U_A$ together with the primes over rational residue characteristics where the prescribed content is nonzero. Both sets are unions of fibres over primes of $L_0$. Taking this restriction and setting all initial division depths to zero proves the required saturation. This gives finite support for each fixed host. A conductor exponent at a supported prime requires its own estimate. Restriction over the fixed field

### How the finite types exclude proper blocks

Here is the cuspidal multiplicity argument for a row with the prescribed good Green types. Write $r = k + m - 1$ and $s = m/2$. The two unitary groups have signatures $(k + s - 1, s)$ and $(k + i, m - i - 1)$ at every real place for $0 \le i < m$. Their finite adelic isomorphism carries the chosen frames and levels. In the image of the finite projector on a passing irreducible constituent choose a single

tensor of finite types $\bigotimes_v \tau_v$ appearing there. The polynomial condition on that tensor gives, for each $1 \le a < r$, a split place $v$ where the degree-$r$ polynomial labeling $\tau_v$ has no divisor of degree $a$. For $2a > r$ use complementary divisors. The place can depend on $a$. Polynomial condition on the selected tuple · Extraction from the actual projector

At this place the radical of the finite parabolic with block sizes $a, r-a$ has zero invariants in $\tau_v$. Indeed the inducing finite cuspidal blocks have the degrees of the irreducible factors of the Green polynomial with multiplicity. In the double-coset formula a block crossing the cut has invariants under a proper radical and contributes zero. The surviving blocks on the first side would give a divisor of degree $a$. In the local group, taking principal-congruence invariants of a representation induced from this parabolic gives finite parabolic induction. This follows by restricting its functions to $\mathrm{GL}_r(\mathcal{O}_v)$ and then reducing modulo the maximal ideal. Compact invariants are exact over $\mathbb{C}$, so every subquotient of the local induction also fails the type test. In particular a representation induced from a proper rational parabolic has zero contribution at the split witness for its first isotropic step. This excludes the residual spectrum for the chosen tuple. Finite and local parabolic argument

For the trace comparison assume stabilization of the discrete trace for these unitary inner forms and the archimedean pseudo-coefficient identities, with matching measures. Compress each finite Hecke function on both sides by the idempotent of the chosen tuple. At the split witness for $a$, averaging the idempotent over the integral radical gives zero. The parabolic descent of every compressed function there consequently vanishes. The proper elliptic endoscopic datum of sizes $a, r-a$ at that split place has exactly this transfer, up to the matching character factors. Every proper term of the stabilized comparison thus vanishes. At infinity the two prescribed discrete series have the same decreasing regular infinitesimal character. The stable elliptic orbital integral of the trace-one pseudo-coefficient is the signed Weyl character expression for that list. At signature $(p, q)$ the sign is $(-1)^{pq} = 1$ since $r$ is odd, and the orbital integral vanishes on nonelliptic regular tori. The strong gap assumption gives the trace selector on the unitary spectrum. The identity datum therefore has equal stable traces on the two forms. Real orbital signs and spectral hypotheses · Stabilization in Kaletha–Mínguez–Shin–White, §3.1

At fixed level and infinity type the contributing cuspidal list is finite. In the common finite Hecke algebra, the nonzero spaces of level invariants of its distinct irreducibles are inequivalent simple modules. The density theorem on their direct sum provides an element acting as the identity on the specified constituent and as zero on all the others. The equality of the two traces gives the difference of the two multiplicities times the rank of the type projector on this constituent. Its rank is positive. This proves equality for the individual passing constituent, including the case of multiplicity zero. The argument uses the polynomial support condition on the same row chosen for the starting datum. Multiplicity separation and scope of the trace argument

### A common normalization of the row periods

The spectral realization requires equality of cuspidal multiplicities on two inner forms for each finite constituent that passes the prescribed level and central idempotent test. The real discrete-series models on the two forms are prescribed at every real place. On the resulting restricted datum, let $K \subset \mathbb{C}$ be the embedded coefficient field and $F_t : A_t \to V$ the period map from the $K$-space for tuple $t$ into the one complex row space. Let $F$ be the complex-linear extension on $(\bigoplus_t A_t) \otimes_K \mathbb{C}$. The row test asks for finitely many complex-linear functionals on $V$ that separate points and take values in $K$ on every $F_t(A_t)$. This is equivalent to the joint kernel of $F$ being defined over $K$. Indeed, composition with separating functionals

gives a matrix over $K$ with that kernel. Conversely, descend the image as the quotient of $\bigoplus_t A_t$ by the descended kernel, choose its basis, and extend the image basis to a complex basis of $V$. Its dual basis supplies the functionals. Cuspidal multiplicity and joint row test · Definition of the separating functionals

The source achieves the common normalization as follows. Fix its reference tuple $t_0$. For each tuple put $C_t = \prod_{i,j} c_{ij,t}$, with every square-comparison constant $c_{ij,t} \neq 0$. The hypotheses give complex-linear separating integral tests $\psi_a$, and, on the good part of each source, the identities

$$\zeta_t C_t \, \psi_a(F_t x) = \kappa \, v_{t,a}(x), \qquad \zeta_t \in K^\times, \quad v_{t,a}(x) \in K.$$

Here $\zeta_t$ is a root of unity and $\kappa$ is common to all the tests and tuples. The finite-group idempotent defining the good part fixes the entire target. Equivariance gives $F_t(ex) = eF_t(x) = F_t(x)$, so the formula applies to any source vector after replacing it by $ex$. The finite expression for $v_{t,a}(ex)$ sums normalized branch pairings over finite classes. Once the local expansions and their coefficient-field comparisons hold, its weights are rational volumes and reciprocals of stabilizer orders, and field operations give the stated rationality. Integral identities and their quantifiers · Finite sum over the actual classes

Each comparison $c_{ij,t}/c_{ij,t_0} \in K^\times$ supplies one factor of $C_t/C_{t_0}$. If $\kappa \neq 0$, multiply every test by the same $C_{t_0}/\kappa$. The value on $F_t(x)$ becomes $v_{t,a}(ex)/(\zeta_t C_t/C_{t_0}) \in K$, and the tests still separate $V$. If $\kappa = 0$, the nonzero factors on the left force every tested value to vanish. The original tests then work and every $F_t$ is zero. This proves the joint descent from the normalized integral identities, including rank zero. The source obtains the square ratios by comparing a definite theta constant and two strings of Hecke $L$-value ratios, with nonvanishing at the integers used as denominators. A full prose construction of this seed still needs those comparisons, the separating tests and their realization on the row with the good finite types used for multiplicity above. Product of square ratios and the zero case · Definite comparison and special-value ratios

## The provisional and final kits on the same host

Choose the constants before the cutoff varies. For fixed sufficiently small $\varepsilon$ and the host constants, choose the lower bounds for the standard record $C$, including the local projection thresholds. Choose $c_D$ above its bound and $c_h \geq 402$ above the bounds that then apply. Take one onset satisfying the host, construction, weight and code requirements. These thresholds are uniform in $x$, the eligible hosts, and the shifts and words at each cutoff. In particular the word bound and the local projection estimates do not call for a separate onset for each word or prime. Recursive quantifiers · Uniform projection thresholds

Use a pair in the nonempty integer windows and one code to build a provisional kit at each supported cutoff. Each kit consists of a point for every shift $0 \leq u_0 < n$. Apply the weight result in its family form, $\mathrm{WClosed}$, to a member at that cutoff. Provenance identifies its scale, rank, $TT$, prime classes and Frobenius coordinates with those of the originating host. The point and host signals agree with the same numerical parameters and orbit representatives. Hence the recovered pair satisfies both weight requirements on that same host. Reapply object existence with this pair, the same host and code, once for every shift. The conditional arithmetic sampling proof above already takes place on the host, and the provisional-kit route is how the release carries it through to the final objects. Both passes of the automorphic construction take place within each application of object existence. Transport from the provisional point · Provisional and final choices · Final kit

**Proved here (one family).** Store the final kit at every $x \in S$ above one threshold $x_1$, with its objects and sizes as $F.D(x,i)$ and $F.L(x,i)$. For formal indexing the release chooses one fixed supported cutoff

$y \geq x_1$ as a fallback. It uses the fallback at $x \leq x_1$, and the index subtype is empty for $x > x_1$ outside $S$. Every supported $x > x_1$ uses its own kit. Within a kit, $\mathrm{Pt.From}$ and $\mathrm{TieSame}$ preserve the ranks, $q, d, QS$, the Artin data, the torus and auxiliary and deformation primes, $TT, T_2$, the weights and code. Consequently every member has $G_\psi$ at its actual scale. The design theorem requests a standard record large enough for the prescribed lower bounds, takes the common onset with the numerical requirements, carries the thick set, tame, $QD$, $TT$ and word tests into the design, and sets $CTu = c_T(\varepsilon, 9)$ with the lower bounds adjusted for that assignment. Its cutoff set is still $S$. Indexing and same-cutoff data Actual scale and retained cutoff set Design construction

## 5. Fitting the comparison and consuming the family

The geometric existence route fixes a class of families and a selection on that class. **Release input (representation and geometric rows).** The input from local Langlands, the unitary group and theta correspondence includes the two theta-integral identities, matching local factors, and the definite-side covolume bound. The source obtains a Fourier coefficient fit and an evaluation grid. It takes the larger of the two grid thresholds for the row functions and for the evaluation theorem and uses that grid for both. The linked row bundles and unramified exponent condition give witnesses for the definite-side hypotheses. $\mathrm{classSel}$ combines the row-existence statements with the lattice-volume identities and the base and retwisting existence statements to obtain one class and one selection satisfying the row requirements together. Its existence proof is $\mathrm{recS}$, using the host-to-design construction of §4 with the recursive input obtained from these rows. In particular the release calls $\mathrm{classSel}$ on a conjunction of the required rows. Choosing a separate family for each row would give a weaker existence assertion. Theta, grid and linked bundles Representation and covolume premises in the final call Common class and selection Simultaneous rows and existence on that class

**Release inputs for the comparison, with the divisor and column constructions supplied through the interfaces of Papers II–V.** On this class the closed row requirements include the common map on the column lattices, the Bessel pairing identification, the rank and fullness assertions, the twisted-place comparison $\mathrm{LinkN}$, and the remaining norm and irreducibility estimates used in $\mathrm{LinkGLOBAL}$. “Closed” here specifies quantifiers. There is a positive cap for $\varepsilon$, then a required bound on the constants for each $\varepsilon$, then a conclusion for every family of the class satisfying that bound. Eventual pointwise conclusions may have an onset depending on that family. The rank input identified as $\mathrm{SmallLevelRankEq}$ equates each selected lift rank $d_{Z,0}$ to $D_0$. The divisibility input $\mathrm{LinkDIVt}$ takes $\mathrm{LinkEta}$ and $\mathrm{DIVinN}$ and supplies $\mathrm{LinkDIV}$. That output includes $B_k(u)(\nu_u) \subseteq CC$ for every word $u$, where $\nu_u$ is its column lattice and $CC$ is the common target lattice, the Bessel comparison by unitary maps and a nonzero scalar, full row lattice, and the stated bounds on the heights and local errors. The common-map theorem and the quantitative divisor, column and norm statements are substantial inputs. Closed quantifiers Common-map definition Bessel, rank and divisibility interfaces Actual comparison inputs

**Proved here (using the length budget).** At each point, $\mathrm{VVle}$ and the identification $\mathrm{sha_eq}$ of its record lengths with the localization lengths give finiteness and bound the nonnegative finite partial sums of $\mathrm{vvTerm}$ by $\sqrt{b}$. Their supremum therefore bounds the infinite sum by $\sqrt{b}$. Let $L = x^{\varepsilon}$, $B_0 = fmbL$, and $N_c \geq 0$ be the size budget of the point. With the selected lift rank identified by $\mathrm{SmallLevelRankEq}$, $\mathrm{LinkN}$ bounds the local sum $\mathrm{etaSum}$ by

$$D_0 \left[ C\left( fmbL^2 + (\log x)^2 N_c \right) + \textstyle\sum_{\ell} \mathrm{vvTerm}(\ell) \right].$$

At a sufficiently late cutoff, $L \ge 1,\ fm \ge 1,\ (\log x)^2 \le L$ and $\sqrt{b} \le b \le B_0$. The first term in the bracket is at most $|C|L(B_0 + N_c)$, and its other term is at most $L(B_0 + N_c)$. This is $\mathrm{DIVinN}$ with constant $|C| + 1$. A maximum of the thresholds for the three inputs and the logarithm estimate makes the deduction uniform over the late members. Thus the localization lengths supply the error allowance in the divisibility bridge. The independent place coverage, column and lattice assertions still enter the bridge. Identification and summability Input inequality and desired bound Pointwise and family deductions

**Proved here (order of choices).** Intersect the cap from design existence with the caps from the closed assertions. After fixing $\varepsilon$ below this positive minimum, join the finitely many required lower bounds on the parameters. Request the single design family at this joined bound. Apply its closed assertions, including the preceding cohomological deduction for divisibility, to this witness $F$. The output $\mathrm{LinkGLOBAL}$ takes $\mathrm{ChosenE}, \mathrm{LinkG}, \mathrm{LinkS}, \mathrm{LinkIRR}, \mathrm{LinkDIV}, \mathrm{LinkMu}$ on $F$, and returns constants and one onset with $\mathrm{ChainTied}$ at every later $x \in S$. Its proof combines the onsets of its input estimates. A tied chain has $D_d.x = x$, chooses all its points from $I_x$, and identifies its arithmetic data, top set, weight windows, and word types with those of $F$. The actual shape call supplies $e_D = 2, e_L = 0$, independent of $\varepsilon$. This proves the assembly theorem of §1 relative to its stated inputs. Closed join and same-family application Actual exponents Global premises and onset Common size threshold Chain ties

**Supplied by Papers II–V (scalar comparison).** The scalar comparison uses the chain and the arithmetic condition at the same member $i$ and the same $TT$. It yields, after enlarging the threshold to cover the chain, arithmetic and scalar onsets,

$$G_z(x) = \sum_{n \le x} \Lambda_{\mathrm{vM}}(n)(x/n - 1) - x \log x + (1 + \gamma_E)x \ \le\ x^{1/2+9\varepsilon} \qquad (x \in S,\ x \text{ sufficiently large}).$$

Here $\mathrm{Level.cyc}(\mathbf{1})$ keeps only the trivial character, so its correction sum over the other kept characters is empty. Given $1/2 < \theta < 1$, choose $\varepsilon$ below the common cap with $9\varepsilon \le \theta - 1/2$ and then take the family. **Proved for ordinary zeta in Paper I, Theorem 1.** The released window criterion uses the resulting bound by $x^\theta$ on arbitrarily late $[Z, Z^{3/2}]$. Its analytic premise, $\mathrm{FormB}$, turns this into zero-freeness to the right of $\theta$ for the kept characters in the release's pole convention. Ordinary zeta is the conductor-one case. The classical zeta functional equation and the given nonvanishing statements give the asserted critical-line conclusion. The full theorem takes this step for general kept characters and then passes through the imprimitive extension. Conductor-one kept set Scalar input uses the same member Window estimate and analytic quantifiers Analytic window premise Final pass to all characters Ordinary critical-line reduction

The favorable-prime refinement of §3.3 can also give a real tail of cutoffs, with the other inputs of this construction. **Proved here from that refinement.** Given a sufficiently large $Z$, use $X = Z^2$ in the conductor-one selection and take $W = Z$. Since $c = k = 1$, §2 supplies $[Z, Z^{3/2}]$, including its left endpoint, so every sufficiently large $x$ belongs to $S$. The releveling, provisional-family and final-family constructions preserve that set. The same-family comparison therefore holds on a full tail and the ordinary all-cutoff Landau consumer applies. The proof of the exported theorem already uses the weaker window assertion. This refinement still needs the pointwise host cohomological estimates and the common scalar, divisor and column inputs. Preserved cutoff set in design All-cutoff analytic interface Prior tail trace

For the favorable ordinary pair, §3.3 supplies the dyadic term on the entire prescribed interval. The general pair-independent route in §3.1 still uses the exceptional-twist $H^2$ estimate. The classical proof in §3.4 supplies the odd minus term for every host in this specialization. In the odd plus term the remaining steps include the deformation and cotangent comparisons, the rank-three attachments in I and V, and the rank-

four automorphic construction in II. The weight selection and signed Hermitian spaces in §4 are on the same arithmetic host. The integral local projection, the normalized mass and norm-window estimates, the saturated automorphic seed with the prescribed types, and the row and spectral comparisons in §§4–5 still enter the geometric construction. Cohomological assembly Existence and comparison assembly