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Paper V. Relative trace, archimedean norms and the rank of period columns

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Paper VI furnishes the selected family at each scale. Paper IV constructs the common integral period maps from its stated inputs. This paper compares those maps to a single complex spectral model, proves the rank and norm implications of the Gram estimates, and derives the archimedean operator-norm inequality from the prescribed metrics. For the relative trace it proves the primitive finite-place count on the selected frame lattice, including at the comparison primes and for Grams with singular reduction. It computes the normalized Euler-product error and derives the signed row-kernel bound from the marked-corner and orbital mass identities. Paper III uses the resulting complex rank and height bounds together with the finite-place estimates. **Proved here** marks the prose proofs. **Release input** marks a pinned theorem used with its indicated hypotheses. **Classical** identifies standard results used in the proofs. Global inputs · Lattice and matrix interface · Finite density details

1. Period maps and arithmetic height

Fix a selected family point (x, i). Let $\mathcal{U}$ be the finite set of words, $P_u : C_u \to H$ the original complex period maps, $d_u = \dim C_u$, and $D_{\mathrm{all}} = \dim H$. Use Hermitian inner products conjugate-linear in the first variable and write $a \otimes b^*$ for $z \mapsto a\langle b, z\rangle$. With π_u the compact stabilizer type, put

$$c_u = \frac{\mathrm{volN}(u)\,\mathrm{volH}(u)}{\dim \pi_u} > 0, \qquad A_u = c_u^{-1/2} P_u, \qquad K_{uv} = A_u^* A_v : C_v \to C_u.$$

Thus $\|K_{uv}\|_{\mathrm{HS}}^2 = \sum_{j,\ell} |\langle A_u e_j, A_v e_\ell\rangle|^2$ in orthonormal bases. Original normalization · Gram coordinates

Integral period and metric inputs. Paper IV supplies the common integral period from its local and global hypotheses. Over the coefficient field L the common row lattice is RR. The L-linear maps $B_u = \mathrm{Bk}(u)$ send the shifted source lattices $\mathrm{Nu}(u)$ into $\mathrm{RR}^\vee$, and these lattices span their respective spaces. At every infinite place w there is a row quotient $p : V_0 \twoheadrightarrow Q$ with its quotient metric, an isomorphism $\iota : Q \to E$ to the common row space, and finitely many subspaces Q_α spanning Q. For all $z \in Q_\alpha, y \in Q$, the metric comparison is $\langle \iota z, \iota y\rangle_E = s_\alpha \langle z, y\rangle_Q$ with $s_\alpha = \kappa(w) \prod_j \lambda_{j,\alpha}(w) > 0$. The base period factors as $B_u(\phi)(\iota p F) = P_u^{\mathrm{base}}(J_m \cdots J_1 \phi)(F)$ for all inputs and rows, where J_ℓ are the prescribed chain maps. The base comparison has scalar one. Common pairing and base normalization · Metric and factorization hypotheses · Lattice inclusion

Proposition 1 (proved here). For bounds $p_u(w) \ge \|P_u^{\mathrm{base}}\|_{\mathrm{op,w}}$ and $k_\ell(w) \ge \|J_\ell\|_{\mathrm{op,w}}$ in the prescribed metrics,

$$\|B_u\|_{\mathrm{op,w}}^2 \le \frac{p_u(w)^2 \prod_\ell k_\ell(w)^2}{\kappa(w) \min_\alpha \prod_j \lambda_{j,\alpha}(w)}. \tag{1}$$

Indeed $\iota^*\iota$ acts by s_α on Q_α by the mixed pairing identity. The subspaces span after scalar extension, so the least eigenvalue is at least $\min_\alpha s_\alpha$. Every quotient vector y has a lift in $(\ker p)^\perp$ of norm $\|y\|_Q$. Evaluate the base factorization on this lift, bound the product of the source maps, and precompose by ι^{-1}, whose norm is at most $(\min s_\alpha)^{-1/2}$. This gives (1) on the whole source space. Operator-norm theorem

In frames X_w, X'_w sending Euclidean into algebraic coordinates, the norm is that of $(X'_w)^{-1} B_u^{(w)} X_w$ with the **dual** frame on the target. For $B_u \ne 0$ the source height is the mean over the infinite places of the totally complex L of $\log \|B_u\|_{\mathrm{op,w}}^2$. The supplied bounds are $p_u(w)^2 \le e^{h_P}$, $k_\ell(w)^2 \le \exp(\sum_{j:\,\ell(j)=\ell} h_j(w))$, and $\mathrm{mean}_w \min_\alpha \sum_j \log \lambda_{j,\alpha}(w) \ge \sum_j l_6(j)$. Taking means in (1) bounds the height by $h_P + \sum_j \mathrm{mean}_w\, h_j(w) - \sum_j l_6(j) - \mathrm{mean}_w \log \kappa(w)$. The comparison to $h(u) = R.h_u - \mathrm{mean}_w \log \kappa(w)$ and the upper bound for $\mu(\mathrm{RR}^\vee) + h(u) - \mathrm{Fo}$ use the geometric chain and arithmetic budgets used by Paper III and supplied in $\mathrm{LinkDIV}$. The zero map satisfies every positive norm envelope. Height definition · Height chain · Height transport · Arithmetic budget

The ordered chain and the common Bessel pairing

Finite-level maps. Let M_u be the complex realization of the ramified starting space, and $e_{\mu,u} : M_u \to \prod_{c \in C_0} \mathbb{C}^{Y_0}$ the starting map at type tuple μ. At μ_0 it is $\mathrm{eMu}_u : M_u \simeq M_u^{\mathrm{ram}}$ followed by inclusion. At $\mu \ne \mu_0$ it is this map followed by $\mathrm{e0Tw}_{u,\mu}$ into the starting function space of the twisted chain. For $j = 0, \dots, m-1$, write c, c' for a source and target lattice class, $D_j^\sharp$, D_{j+1}^- for multiplication by the determinant powers $\prod_\tau d_{j,\tau}^{s_{j,\tau}^\sharp}$, $\prod_\tau d_{j+1,\tau}^{-s_{j,\tau}^{\mathrm{div}}}$, and $U_{j,\mu}(c,c')$ for the unfolded theta map on the indicated function spaces. The source defines

$$(K_{j,\mu} f)_{c'} = \sum_c \frac{\rho_{j,\mu}(c)\rho'_{j,\mu}(c')}{|\mathrm{Stab}_j(c)|} D_{j+1}^- U_{j,\mu}(c,c') D_j^\sharp f_c, \qquad f_m = K_{m-1,\mu} \cdots K_{0,\mu} e_{\mu,u} \phi.$$

The terminal pair is $\widetilde{P}_{u,\mu}(\phi, F) = \mathrm{period}_{u,\mu}(\mathrm{lowerMap}_{u,\mu}\, \mathrm{evz}_{u,\mu}\, f_m, F)$ in matched row coordinates. The row argument in the terminal pairing is $\mathrm{eMt}_u^{-1} F$ at base and $\mathrm{eMtTw}_{u,\mu}^{-1} F$ at a twisted tuple. With these row arguments the release identifies $\widetilde{P}_{u,\mu}$ with $\mathrm{perCol}(u, \mu, \cdot, \cdot)$ at both levels. On the rational

column face, $\mathrm{Pt}_{u,\mu}(x,y)$ is the corresponding bilinear map obtained by taking the ramified basis on input and the specified degree-zero evaluation map on rows. Its embedding agrees with the ordered period on those inputs. In particular at base the comparison is on every rational basis pair, with no scalar in this coordinate identification. Ordered kernels · Base maps · Twisted maps · Terminal maps · Rational column face · Base coordinate equality

Release input (seesaw). The comparison uses compatible families of definite lattice kernels and indefinite Bessel and Fourier–Jacobi models, with common Weil data, the chosen quotient measures, and the local level and support laws. One substantive equality is the terminal two-step identity. Write t for a tuple in the family, $j = m-2$, S_j for the specified map on ramified columns in that identity, and $\phi_\alpha = \mathrm{rd}_{j,\alpha}(\phi, y)$ for the read at frame variable y in class α. For each allowed pair of polynomials P_1, P_2 there is **one** real two-step datum $\mathcal{C}$ for all t. With the two prescribed upward-in-index row lifts written as $L_j^{P_1}$ and $L_{j+1}^{P_2}$, the identity is

$$\mathrm{perF}_{t,j}((\phi_\alpha)_\alpha, L_j^{P_1} L_{j+1}^{P_2} f) = c_V(t,j)\,\mathrm{perT}_{t,m}\big(\mathrm{two}_{t,j}((\mathrm{rd}_{j,\alpha}(S_j\phi, y))_\alpha, \mathcal{C}), f\big). \tag{1a}$$

Here f is a terminal test from the prescribed multiplicity space evaluated on a smooth tensor. On the left perF is the sum over α of Bessel integrals against finite weighted right translates of its row argument. On the right two sums over the source and intermediate classes α, γ for each final class β, and perT sums the terminal integrals $\int_{\mathcal{F}_T} (\mathrm{trc}(e_{T,\beta}) f)(h)\, f_\beta(\mathrm{emb}_T(h,1))\, d\mu_T(h)$. The definition of $\mathrm{trc}(e)$ is $\sum_g e(g) R(g)$, with $R(g)f(h) = f(hg)$. The proof expands the two lifts inside the Bessel integral, uses the disjoint supports of the places serving the finite modifications, and applies the local collapse and the common-real-datum contraction. In the terminal integration the common datum is obtained by applying the real contraction operator to the two vacuum inputs. Independence of the auxiliary flat representative is used to compare this datum between tuples. Two-step and terminal definitions · Terminal seesaw statement · Unfolded integrals · Common datum and assembly

Adelic dictionaries. In the spectral model the column of a seed T is the finite-level map

$$\mathrm{colDict}_u(T)(c,y) = J_u^{\mathrm{val}}(\xi_{T,c})(e_Y y), \qquad \langle \xi_{T,c}, \xi \rangle = T(\xi)(1, h_c) \quad (\xi \in \pi_u).$$

Here h_c represents the finite stabilizer class and J_u^{val} is the conjugate-linear compact evaluation map. The two conjugate-linear operations make the dictionary linear. The matched level identifies K_u with the finite column level after conjugation by the specified level change h_0, and matches their characters. The class decomposition is $h = \gamma h_c k$, $k \in K_u$, with a unique class c. In the tied model this dictionary is bijective onto the full column space, preserves its metric with the chosen column norm, and conjugates the local diagonal right translations into the actions used to impose the ramified conditions. The isometry that transports the original P_u to the concrete Poincaré map has this same column dictionary as its inverse coordinate identification. These are **release inputs**. Evaluation and Riesz dictionary · Level and character matching · Isometry and diagonal action · Pinned inverse

On rows, write ι_X for the group identification and t_S for the chosen translation. A fixed tensor of local real vectors v_0 gives the translated evaluation of a row multiplicity vector R, with automorphic function $f_R = R(\bigotimes_w v_{0,w})$. Its L^2 representative is built from $g \mapsto f_R(\iota_X^{-1}(g) t_S)$. The row isometry $j : E_{\mathbb{C}} \to EEC$ identifies the common row space with its spectral realization, and the ambient isometry identifies $s_E F$ with $\mathrm{conjHH}(jF)$, where $s_E : E_{\mathbb{C}} \to H$ is the original conjugate-linear isometry into H. A reference normalization in the join compares the two row evaluations by a single nonzero scalar, expressed as $\kappa_S \kappa_0$ in the construction. This comparison holds on all common rows. The construction

restricts to the chosen good-level row space. Translated evaluation · Common row identification · Nonzero evaluation factor

For explicit measures, define the adelic Bessel functional in the matched decomposition by

$$\mathscr{B}_u(\varphi, f) = \int_{\mathcal{F}_{H_u}} \int_{\mathcal{F}_{N_u}} \varphi(h) f(n\,\mathrm{emb}_u(h))\,\overline{\psi_{\mathrm{Bes}}(n)}\,d\nu_{N_u}(n)\,d\mu_{H_u}(h).$$

The pinned character is the one with parameter 1 and the chosen distinguished line in BesLev. With $\mathrm{colForm}_u(T,\xi)(h) = T(\xi)(\mathrm{toH}_u(h))$, the unfolding for the L^2 representative of a bounded continuous row function f with matching rational invariance and inverse level character gives

$$\left\langle \overline{f}, P_{u,T} \right\rangle = c_u^{\mathrm{Haar}} \int_{G_\infty} \mathscr{B}_u(\mathrm{colForm}_u(T, W_u(y)),\ g \mapsto f(\iota_X(g)(y, x_u)))\ d\nu_{Q,u}(y). \tag{1b}$$

Here $\nu_{Q,u}$ is the release real quotient measure represented on G_∞, and x_u is the fixed finite translate of the seed. The release first unfolds to the rational stabilizer fundamental domain, then disintegrates over the Bessel quotient and y. The factor c_u^{Haar} is positive by this Haar comparison. The compact quotient and the bound by a constant times $\|W_u(y)\|$ justify the integrations. The complex scalar relating the ordered chain to the spectral pairing below also includes the normalization of the row evaluation and the finite comparison. Bessel functional · Parameter · Level and column form · Inner product unfolding · Positive Haar factor · Measurable disintegration and L^1 bound

Nonvanishing and the common comparison. The terminal identity and the earlier Fourier–Jacobi steps feed a joint statement on the selected package. It chooses one finite central translation z for all u. Its word data give a source map of norm $r_u^{\mathrm{sc}}\|\phi\|$ with $r_u^{\mathrm{sc}} > 0$, and factors $c(u,\mu) \neq 0$ such that, when $c(u,\mu(w))/c(u,\mu_0)$ comes from the coefficient field, its preimage has valuation zero at w. The matching transports two-term linear relations among the normalized finite chain periods to the bottom Bessel periods, and thence to the spectral pairings at zt_S. The bottom group alignment uses the same z for the wordwise finite transport corrections and allows a right factor in the common row level. These are **release inputs** on the bottom, family and served-place hypotheses. Joint condition and bottom transport · Preservation of two-term relations · Selected assembly

The constants in the repeated comparison can be read from the ordered recursion. Write $n = m/2$. In the tuple notation, the forward column recursion in the two-step proof applies $K_{2i+1}K_{2i}S_{2i}$ at step i. The backward row recursion composes the chosen maps io_j. The total factors in this comparison are

$$c_{\mathrm{top}}(t) = c_P(t) \prod_{i=0}^{n-1} c_V(t,2i)c_K(t,2i)c_K(t,2i+1), \qquad c_{\mathrm{bot}}(t) = \prod_{j=0}^{m-1} c_I(t,j).$$

The comparison of terminal coordinates supplies a nonzero word factor s_u and a column similitude S_u. The source uses $c(u,t) = s_u c_{\mathrm{top}}(t)/c_{\mathrm{bot}}(t)$ and the column change S_u^{-1}. The c_I are nonzero by the row lift laws, $c_{\mathrm{top}} \neq 0$ is part of the tied factor comparison, and the ratio to the base cancels s_u. The unit-ratio theorem multiplies the two-step ratios $c_V c_K c_K/(c_I c_I)$ and the terminal factor ratio. The served-place condition is used to deduce valuation zero for an algebraic preimage of that ratio at the coefficient place in question. Ordered recursions and factors · Factor ratios · Nonzero factors and linear relations

The nonzero reference pairing is obtained independently of the integral row comparison. The source uses a same-word defect bound $\|K_{uu} - I\|_{\mathrm{HS}}^2 \leq e\,d_u$ and a retained dimension $> (e\,d_u)$, supplied in the

selected range by the earlier diagonal and ramified rank estimates. Indeed each vector of an orthonormal basis of $\ker P_u$ contributes one to the squared defect, since K_{uu} kills that vector. Thus a subspace of dimension $> ed_u$ cannot lie in the kernel. The resulting assertion PuNZ gives a ramified vector with $P_u\phi \neq 0$ for every word, with ϕ understood in full column coordinates. At the common translated level the conjugate row evaluation is onto the spectral row space used for this step. Taking the inverse image of that nonzero Poincaré vector gives a test with $b_0 = \langle P_{u,T}, P_{u,T}\rangle > 0$. This step uses the surjectivity of the full conjugate row evaluation for the fixed vector v_0. It precedes the arithmetic projection in the Gram argument. Kernel and diagonal bound · Selected nonvanishing · Reference pair and its value

Proved here (comparison from linear relations). Fix the nonzero reference value b_0 for a word u. Let p_0 be its reference scalar on the chain side of the relation law, and let $L = c(u,\mu)\widetilde{P}_{u,\mu}(\phi, F)$ be a tested chain value. The law states that $\alpha p_0 + \beta L = 0$ implies $\alpha b_0 + \beta b = 0$ for the matched spectral value b, after the stated word change of columns. Taking the relation with zero second coefficient gives $p_0 \neq 0$, since otherwise $b_0 = 0$. Taking $(\alpha, \beta) = (L, -p_0)$ gives $Lb_0 = p_0 b$. Thus the comparison scalar is independent of the test. In transported coordinates, with a common choice of embedding or conjugate embedding σ_0, the source packages these identities as

$$\zeta^{\mathbb{C}}_{u,\mu}\,\sigma_0(\mathrm{Pt}_{u,\mu}(x,y)) = \alpha_u \left\langle s_\infty(V_\infty \Theta_\mu j_{R,\mu} e_{V,\mu} y), P^{\mathrm{ind}}_u(U_{\infty,u}\sigma_0(x))\right\rangle . \tag{1c}$$

Here $e_{V,\mu}$ gives the algebraic rows, $j_{R,\mu}$ their complex realization, Θ_μ their map into the common good-level row space, and s_∞ is conjugate-linear. The single V_∞ is the common translation on rows, and the $U_{\infty,u}$ are the isometries after division by the positive source norm scales. At the normalization step the source takes $\zeta^{\mathbb{C}}_{u,\mu} = c(u,\mu)/c(u,\mu_0)$, $\alpha_u = p_0/(b_0 c(u,\mu_0))$, where p_0, b_0, c denote the factors in the transported coordinates of the anchored statement. All denominators are nonzero, and $\zeta^{\mathbb{C}}_{u,\mu_0} = 1$. Linear-relation proof · Transport of the anchors · Normalization and nonvanishing · Packaged identity

The scalar convention in (1c) puts $\zeta^{\mathbb{C}}$ on the *chain* side. With $R_\mu = \mathrm{rowMat}_\mu P_{q,\mu}$ the matched common row map, the source proves $\zeta^{\mathbb{C}}_{u,\mu}\sigma_0(\mathrm{Pt}_{u,\mu}(x,y)) = \sigma_0(B^{\mathrm{def}}_u(x)(R_\mu y))$. The multiplier of the common period in Paper IV's convention is the reciprocal of the chain-side scalar. Its valuation is also zero whenever the chain-side scalar is a unit. If some period value at the tuple is nonzero, the displayed equality realizes $\zeta^{\mathbb{C}}_{u,\mu}$ as the embedding of a quotient in the coefficient field. If the scalar has no preimage, all the period values at that tuple vanish, and the rational comparison can use 1. The base chain kills the kernel of P_{q,μ_0} because the right side vanishes on it. This explains the section formula $B^{\mathbb{C}}_u(\phi, F) = \mathrm{perCol}(u, \mu_0, \phi, \mathrm{sec}_{\mu_0} F)$ in the record. Here sec_{μ_0} inverts the row map on the orthogonal complement in the degree-zero space of its kernel there. Finite tuple comparison and kernel · Units and absent preimage · Degree-zero section

Release input (Rallis and doubling). These supply the preliminary lift and norm theorems used to construct the comparison. For source A and target B with $a = \mathrm{rank}\, A < b = \mathrm{rank}\, B \le 2a$, the convergent source statement assumes B anisotropic or $a + \mathrm{Witt}(B) < b$. It further assumes the compatible doubled Weil data, the covolume-one polarization measure, the vanishing of the lower lifts on the chosen cuspidal representation, and the specified compact-finiteness of the doubled Schwartz vector. In the source convention set $s_0 = (b-a)/2$. The test section at s_0 is $\Phi_{\mathrm{SW}}(g) = \int_{\mathcal{F}_{\mathrm{pol}}} (\omega^{\mathrm{dbl}}(g,1)\delta(\Phi_1,\Phi_2))(\ell w)\, dw$, with $\delta(\Phi_1,\Phi_2)(x) = \Phi_1(\mathrm{res}_L\, x)\overline{\Phi_2(-\mathrm{res}_R\, x)}$ and ℓ the polarization map. The convergent Rallis identity says the doubled zeta value paired with f_1, f_2 at s_0, for the flat section through Φ_{SW} and inverse character χ_B^{-1}, equals

$$\frac{\langle \theta_{\Phi_1} f_1, \theta_{\Phi_2} f_2 \rangle_{\mathrm{Pet,B}}}{\mathrm{vol}([U(B)])}$$

in the release Petersson convention. The theorem includes convergence and L^2 of the two lifts. The source obtains the lift inner products in both orientations via the unitary Weil–Siegel formula and Eisenstein continuation. This inner-product theorem supplies norms and surjectivity inputs for the comparison. The equalities of linear functionals in (1a)–(1c) use the seesaw and its transported linear relations. Doubled vector, section and point · Convergent Rallis statement · Both orientations

The specific surjectivity input in the family is $\mathrm{OntoAll}$. Under the node gap and the chosen step conditions, every reached composition of maps on the selected multiplicity spaces, from stage $k \le m$ to stage zero, is onto. The source derives it from the stated Howe, Weil representation, convergent Siegel–Weil and doubling inputs, then invokes it in constructing the family comparison. For the local zeta nonvanishing step, on $b\kappa$ with b real and κ in the finite level, the conjugate level characters cancel. The integrand becomes the real Weil coefficient times the conjugate real matrix coefficient times $\overline{\mathrm{ip}(q,q)}$, where $q \neq 0$ is the multiplicity vector. Outside the level it vanishes on the prescribed integration domain. Haar disintegration then carries the nonzero real integral to the adelic one, since $\mathrm{ip}(q,q) > 0$. The actual real test and its integral, the unramified doubling factors, the normalizations at modified finite places, and the global doubling continuation, Weil–Siegel and lift-surjectivity theorems remain inputs here. In particular the release's proposition named $\mathrm{DoublingMethodLocalZetaIntegrals}$ includes the global flat-section basic identity in the right half-plane and continuation. The symbols $\zeta_{u,\mu}$, α_u in (1c) come from the matched period comparison. Their values in the rank argument are fixed by the period identities and nonvanishing above. Surjectivity statement · Surjectivity from theta and doubling · Family use of doubling · Local zeta nonvanishing · Doubling input name · Scope of doubling input

Proved here (one projection for all pairs). In the join, let $\jmath_R$ be the common map denoted by e_R in the source, taken with its one fixed conjugation choice ϵ. The row comparison intertwines $\jmath_R$ and V_∞ through a common similitude. Pullback gives one complex-linear isometry $V_R = \jmath_R^{-1} V_\infty \jmath_R$ on the common rows because the positive similitude factor cancels in its norm. If $c_{\phi,u} : M_u \to \mathbb{C}^{r_u}$ is the column coordinate map, the pulled-back column map is $c_{\phi,u}^{-1} \epsilon U_{\infty,u} \epsilon c_{\phi,u}$, with ϵ acting entrywise. The map $\epsilon c_{\phi,u}$ scales norms by one positive constant in the prescribed column frame. That constant cancels after pullback and the two occurrences of ϵ make the result complex-linear. This gives the isometry U_u of M_u, using the pinned inverse dictionary to identify its target. In the source equation $\Theta_{\mu_0} \jmath_R e_V(y) = \lambda\, \jmath_R(\iota(c_F y))$ the row scalar $\lambda \neq 0$ is common. Because the row realization is conjugate-linear and the inner product conjugates the first slot, λ appears as λ in the bilinear finite expression. Substituting the tied Bessel pairing with comparison factor $\kappa_u \neq 0$ gives $a_u = \epsilon(\alpha_u \lambda)\kappa_u \neq 0$. Equality on the rational coordinate basis pairs suffices to extend to all complex inputs by bilinearity. The base row map is onto the common rows by the join theorem, so for any row F one can choose its degree-zero lift in the section formula. With the pinned column dictionary, this yields for all u, ϕ, F

$$B_u^{\mathbb{C}}(\phi, F) = a_u \langle s_E(V_R F), P_u(i_u U_u \phi) \rangle, \tag{1d}$$

The isometric inclusion is $i_u = (s_{C_u})^{-1} \circ \mathrm{incl}_{M_u^{\mathrm{ram}} \subseteq M_u^0} \circ \mathrm{eMu}_u$, with s_{C_u} the original coordinate equivalence onto the full initial column space M_u^0. At rational inputs, $B_u^{\mathbb{C}}(j_{N,u} x, j_E y) = \sigma(B_u(x)(y))$; this is the separate rationality tie used for Bk. Row, column and pairing ties · Transport and common scalar · Section and original Bessel map · Rationality tie · Isometric inclusion · Surjectivity of the base row map

Set $H_{\mathrm{rows}} = s_E(V_R E_{\mathbb{C}})$ and let Π be its orthogonal projection. The same orthonormal basis of H_{rows} works in (1d) for every u. With $V_u = i_u U_u$ in orthonormal source coordinates and the common row basis, the matrix in (1d) is $a_u[\Pi P_u V_u]$. The source implements this as the row-basis transpose, the embedded rational matrix, and a wordwise source-basis change multiplied by a_u^{-1}. Passing from the chosen embedding to the representative of its infinite place can conjugate all matrices at once. In the chosen embedding, the Gram blocks of these projected columns have factors $\overline{a_u} a_v \sqrt{c_u c_v}$ times $V_u^* A_u^* \Pi A_v V_v$. These factors come from one scalar per word. The unprojected spectral bound applies to the synthesis map before projection, and the rank estimate in §5 charges the row codimension after projection. The same map gives $\|B_u\|_{\mathrm{op}} \le |a_u|\sqrt{c_u}\,\|A_u\|_{\mathrm{op}}$ in the tied metrics at this embedding. Equation (1) supplies the separate all-place height bound. The family selector uses the same join, multiplicity package and orthonormal row realization at each (x, i) for every word. These choices can depend on that point. Orthonormal row extension · Exact rational block and scalars · Simultaneous selection · Insertion in the division package

2. Relative trace identity

Let $\Gamma = G(F)$ be the rational unitary group, where F is the totally real base field. Let Γ_u, Γ_v be the rational stabilizers of the two seeds. A concrete input T gives a seed $\Phi_{u,T}$ supported on the cell $g_f = nhx_u k$, $k \in K_2$, with value $\psi_u(n)\chi_2(k)T(W_u(g_\infty))(1,h)$. The concrete period map sends T to the L^2 Poincaré series $P_{u,T}(g) = \sum_{\alpha \in \Gamma_u\backslash\Gamma} \Phi_{u,T}(\alpha g)$. In this model the positive constant is

$$c_u^X = \mathrm{vol}(K_2)\,\mu_{N_u}(F_{N_u})\,\mu_{H_u}(H_u)/\dim \pi_u.$$

The isometric model comparison preserves the corresponding normalized constants. Seed and series · Concrete pairing and volume · Model comparison

Proposition 2 (unfolding, proved here given the release inputs). Under absolute convergence, with the quotient measures of the model,

$$\begin{aligned}\frac{\langle P_{u,T}, P_{v,T'}\rangle_{L^2}}{\sqrt{c_u^X c_v^X}} &= \frac{1}{\sqrt{c_u^X c_v^X}} \sum_{\gamma \in \Gamma_u\backslash\Gamma/\Gamma_v} \int_{(\Gamma_u \cap \gamma\Gamma_v\gamma^{-1})\backslash G(\mathbb{A}_F)} \overline{\Phi_{u,T}(g)}\,\Phi_{v,T'}(\gamma^{-1}g)\,dg \\ &= \Big\langle \Big(\sum_t a_t\Big) J_v T', J_u T \Big\rangle,\end{aligned} \tag{2}$$

where $J_u = \mathrm{sharp}$ is the conjugate-linear Riesz map to the direct sum of compact type spaces over the finite column classes. Expand the first series and translate each summand to unfold the integral to $\Gamma_u\backslash G(\mathbb{A}_F)$. In the remaining series the right Γ_u-orbit of $\Gamma_v\gamma^{-1}$ is indexed by $(\Gamma_u \cap \gamma\Gamma_v\gamma^{-1})\backslash\Gamma_u$. Unfold that orbit by the same translation. Absolute convergence justifies these exchanges. The second equality uses the **release input** of typed finite reduction and compatible fundamental domains for the finite slabs. Unfolding hypotheses · Exact kernel expansion

In the reduction, $t = (\Lambda, \Lambda', [\gamma])$ ranges over the two sets of selected lattices and double classes of rational isometries carrying Λ' to Λ and satisfying the lifted-flag condition, modulo their flag stabilizers $N_\Lambda, N_{\Lambda'}$. Define Ω_t to be the real fundamental domain for the image of $N_\Lambda \cap \gamma N_{\Lambda'}\gamma^{-1}$. The integral in each surviving block is

$$O_t = \int_{\Omega_t} W_u(g) \otimes W_v(\gamma_\infty^{-1} g)^*\, d\mu_\infty(g). \tag{3}$$

For column classes ρ, ρ', the block is zero unless the respective lattices match and the transported finite element $k(\rho, \rho', t)$ belongs to K_2. In a surviving slab the two copies of the level character cancel as functions of the integration variable. Integrating the remaining fixed phase gives

$$a_t(\rho, \rho') = \frac{\mathrm{vol}(K_2)}{\sqrt{c_u^X \, c_v^X}} \, \chi_2(k(\rho, \rho', t)) \, O_t. \tag{4}$$

Evaluation by the column functionals converts the rank-one integrand into the pairing in (2). The reversed inner product in (2) accounts for the conjugate-linear Riesz maps. Their isometric transports, including the common ambient and wordwise source isometries, give $\|K_{uv}\|_{\mathrm{HS}}^2 = \|\sum_t a_t\|_{\mathrm{HS}}^2$ in tied models. Rational term index · Real domain · Finite blocks · Slab integration · Hilbert–Schmidt transport

3. Regular real orbits and their energy

Fix the same concrete setting and word pair. Put $G_\infty = \prod_{w\mid\infty} G(F_w)$, $r =$ flag length and $N = 2r + k + 1$. Form $Z(g)$ at each real place by pairing the first ordered frame against the translate of the second. At scale $\mathrm{sc} > 0$ the frame entries carry factors $\mathrm{sc}^{-(r-j)}$ for indices $j = 0, \dots, r$. The selected application uses $\mathrm{sc} = 1$. For a diagonal representative $h(c)$ with entries $c_1, \dots, c_{r+1}$, write $D_j = \prod_{a\le j} c_a$, $D_0 = 1$, and

$$\Delta = |c_{r+1}|^2 - TT', \qquad G_v = |D_{r+1}|^2 - TT'|D_r|^2 = |D_r|^2 \Delta,$$

where $T, T' > 0$ are the last-vector norm parameters. When the early minors are nonzero, diagonalization leaves r hyperbolic pairs and a two-vector Gram with determinant $TT' - |c_{r+1}|^2$. The section realizes the locus $\Delta > 0$ by a Witt frame, with a last component proportional to $\sqrt{|c_{r+1}|^2/T - T'}$. Frame and invariant definitions · Witt construction · Section matrix

Release input (real Haar disintegration). Let S, S' be the two real stabilizers with Haar measures coming from their unipotent and compact factors. The good set is the product of the placewise sectors $c_j \neq 0,\ \Delta > 0$. With a measurable diagonal section h and $C_{\mathrm{oo}} > 0$, for every nonnegative measurable f,

$$\int f \, d\mu_\infty = \int_{\mathrm{Good}} \int_S \int_{S'} f(s h(c) s'^{-1}) \, d\mu_{S'} \, d\mu_S \, dm(c), \qquad dm = C_{\mathrm{oo}} \prod_w \left(G_{v,w}^{k-1} \prod_{j=1}^{r} |D_{j,w}|^4 \right) dc_\infty, \tag{5}$$

Here $dc_\infty = \prod_{w,j} 2 \, d\Re c_{j,w} \, d\Im c_{j,w}$. The local Jacobian theorem gives a full-measure regular chart, excluding the vanishing early minors and residual Gram determinant. The real geometry input also includes the stabilizer double-coset comparison and unimodularity. In particular the chart is a statement about ambient Haar measure. A discrete rational sum can meet the exceptional sets. Measure and disintegration · Local-to-global Haar theorem · Witt comparison

Proved here. The complex Jacobian of $c \mapsto (D_1, \dots, D_{r+1})$ is $\prod_{j=1}^{r+1} D_{j-1}$ by triangular differentiation. The placewise density in minor coordinates is consequently $(|D_{r+1}|^2 - TT'|D_r|^2)^{k-1} \prod_{j=1}^{r} |D_j|^2$ against the same product of twice planar measures, times the constant in (5). In c-coordinates the factor is $\Delta^{k-1} \prod_{j=1}^{r} |c_j|^{2(k-1)+4(r+1-j)}$. These give the measure weights at the residual-Gram boundary and at early-minor faces. Jacobian and invariants · Density

The real energy uses full Haar:

$$O(g)=\int_{G_\infty} W(z)\otimes W'(g^{-1}z)^*\,d\mu_\infty(z),\qquad I_{\mathrm{oo}}=\int_{\mathrm{Good}}\|O(h(c))\|_{\mathrm{HS}}^2\,dm(c).\tag{6}$$

Write $d_\sigma=\prod_w d(\sigma_w,\mu_w)>0$ for the ambient formal degree, and w_0 for the tensor of unit lowest-type vectors. The **release inputs** are continuous packets, unitary left equivariance $W(sz)=\tau(s)W(z)$, the left quotient measures ν,ν' for the same μ_∞, and

$$\int\|W\|\,d\nu<\infty,\quad \int\|W\|^2\,d\nu=1,\quad W(x)=d_\sigma\int\overline{\langle w_0,\sigma(y)w_0\rangle}\,W(xy)\,d\mu_\infty(y).$$

They include the primed versions, Bochner integrability in (6) on every good fibre, and finiteness there of

$$B(c)=\int|\langle\sigma(z)w_0,\sigma(sh(c)s'^{-1}z')w_0\rangle|\,\|W(z)\|\|W'(z')\|\,d\nu(z)\,d\nu'(z')\,d\mu_S(s)\,d\mu_{S'}(s').\tag{7}$$

Two finite triple majorants use one packet twice and one stabilizer variable, replacing $sh(c)s'^{-1}$ by s. The stabilizers admit measurable Følner sets with positive finite Haar mass and normalized left overlaps tending to one for every fixed compact subset. Wave and energy assumptions · Local transport and convergence · Product convergence · Følner sets

The concrete convergence theorem uses the discrete-series Harish–Chandra parameter, its lowest-type vector, the two positive last-vector norms, side types, and weighted matrix-coefficient bounds. The arithmetic hypothesis is $N\ge 1801$ with the specified adjacent gaps at least $h=16N^{19}$. At a place the bound for (7) is a fixed constant times $\int\|s\gamma s'^{-1}\|_F^{-\beta}\,d\mu_S d\mu_{S'}$, with $\beta=h/(8\sqrt{2(r+1)})-4(k+r)(r+1)-1>9(r+1)N^2$. On regular fibres the norms of both unipotent factors are bounded by a constant times a power of the translated matrix norm. This uses inverses of early minors. The local representation estimates and the uniformity on compact subsets of the good locus are used in their source forms. Weight gap · Decay exponent and integral · Unipotent bounds and Haar integrability · Regular section

Proposition 3 (proved here). Under (5), the wave-packet hypotheses, and Schur orthogonality, $I_{\mathrm{oo}}\le 1/d_\sigma$.

Classical. With $d=d_\sigma$, Schur orthogonality for the square-integrable representation and the chosen Haar measure says

$$\int\langle b,\sigma(g)a\rangle\overline{\langle b',\sigma(g)a'\rangle}\,d\mu_\infty(g)=d^{-1}\langle a',a\rangle\langle b,b'\rangle.$$

The product formula for d follows on tensors by Fubini and extends by the same square-integrable coefficient identity. We take the discrete-series identification and this normalization on the selected group from the release. Formal-degree hypothesis · Selected product identity

For a Følner set $\Omega\subset S$ of mass a, define

$$Y_\Omega e=\int\!\!\int_\Omega\langle W(t^{-1}z),e\rangle\,\sigma(t^{-1}z)w_0\,d\mu_S(t)d\nu(z).$$

The vector integral converges by the quotient L^1 assumption and unitary equivariance. Use the primed version with mass a'. Expanding the sum of squared norms of $Y_\Omega e_j$ in an orthonormal basis and substituting $q=t(t')^{-1}$ for the two copies of the S-variable gives

$$\frac{\|Y_\Omega\|_{\mathrm{HS}}^2}{a} = \int \frac{\mu_S(\Omega \cap q^{-1}\Omega)}{a} \langle \sigma(z)w_0, \sigma(qz')w_0\rangle \langle W(qz'), W(z)\rangle \, d\mu_S(q) d\nu(z') d\nu(z).$$

In this calculation the group action $\tau(t)$ cancels between the two packet vectors and $\sigma(t)$ cancels between the two representation vectors. The ratio is at most one and tends to one for each q along the Følner sequence. Domination follows from the triple majorant. In the limit the quotient formula combines q, z' into $x = qz'$. The reproducing identity at z, conjugated in the first slot of $\langle W(x), W(z)\rangle$, reduces the x-integral to $\|W(z)\|^2/d$. Hence $\|Y_{\Omega_n}\|_{\mathrm{HS}}^2/a_n \to 1/d$, and likewise on the primed side. Truncated wave packets · Overlap and norm limit

Set $X_n(g) = d\, Y_{\Omega_n}^* \sigma(g) Y'_{\Omega'_n}$ and

$$F_n(c) = \frac{\int_S \int_{S'} \|X_n(sh(c)s'^{-1})\|_{\mathrm{HS}}^2 \, d\mu_{S'} d\mu_S}{a_n a'_n}.$$

In orthonormal bases each matrix entry of $X_n(g)$ is $d\langle Y_{\Omega_n} e_i, \sigma(g) Y'_{\Omega'_n} e'_j\rangle$. Squaring and summing the Schur identity gives $\int \|X_n\|_{\mathrm{HS}}^2 \, d\mu_\infty = d\|Y_{\Omega_n}\|_{\mathrm{HS}}^2 \|Y'_{\Omega'_n}\|_{\mathrm{HS}}^2$. By (5), $\int F_n \, dm \to d \cdot d^{-1} \cdot d^{-1} = d^{-1}$. Schur computation

Fix now $\gamma = h(c)$. The kernel whose norm is dominated by d times the integrand of (7) is

$$\mathcal{K}_\gamma(z, z', t, t') = d\, \langle \sigma(t^{-1}z)w_0, \sigma(\gamma(t')^{-1}z')w_0\rangle W(t^{-1}z) \otimes W'((t')^{-1}z')^*.$$

Its full integral in z, z', t, t' is $O(\gamma)$. In fact the left quotient formulas and Haar inversion turn the integral into a double ambient integral in $x = t^{-1}z,\ y = (t')^{-1}z'$. Integrating x first reproduces $W(\gamma y)$ since $\langle \sigma(x)w_0, \sigma(\gamma y)w_0\rangle = \overline{\langle w_0, \sigma((\gamma y)^{-1}x)w_0\rangle}$. Changing γy to x gives (6). Absolute convergence follows from (7) because the Hilbert–Schmidt norm of $a \otimes b^*$ is $\|a\|\|b\|$. Lifted kernel and reproducing integral · Fourfold form

Expanding $X_n(s\gamma s'^{-1})$, changing the subgroup variables and removing the unitary type actions gives the integral of $\mathcal{K}_\gamma$ with masks $1_{\Omega_n}(ts^{-1}) 1_{\Omega'_n}(t'(s')^{-1})$, with the same Hilbert–Schmidt norm. In the outer variables s, s', use the measure $d\mu_S d\mu_{S'}/(a_n a'_n)$. Each mask has L^2 norm one by Haar invariance and inversion. Its L^2 distance to the mask with $t = t' = 1$ tends to zero for each t, t' by the normalized overlap property, and is bounded by two. Minkowski's integral inequality followed by dominated convergence in the four kernel variables shows that the two masked integrals differ by a term tending to zero in outer L^2. The reference masked integral has outer L^2 norm $\|O(\gamma)\|_{\mathrm{HS}}$. This proves pointwise $F_n(c) \to \|O(\gamma)\|_{\mathrm{HS}}^2$. Fatou gives (6) bounded by $1/d$. Truncation and translated masks · Pointwise bound and Fatou in the release

For the boundary of the real integral, the same Minkowski estimate gives $F_n(c)^{1/2} \le dB(c)$. The selected coefficient $f(g) = |\langle w_0, \sigma(g)w_0\rangle|$ is Haar integrable by the **release input** of weighted discrete-series coefficient integrability. Tonelli, (5), and both Haar translations give $\int B \, dm = \|f\|_1 \|W\|_{L^1(\nu)} \|W'\|_{L^1(\nu')}$. The first moment stays finite across the boundary. An equality $I_{\mathrm{oo}} = 1/d$ by dominated convergence on the whole base would follow from an $L^2(m)$ bound for B. The inverse-minor bounds deteriorate near the early-minor faces. The proof of the required upper bound uses only Fatou. Coefficient integrability · Product coefficient · Pointwise convergence input · Energy conclusion

Removing the real cutoff on the chosen periods

Fix a pair of words at one arithmetic cutoff. At a real place let D_j denote the leading minors of their cross Gram matrix, r the flag length and T, T' the two positive norms. The invariant used to smooth the orbital integral is

$$G = |D_{r+1}|^2 - TT'|D_r|^2.$$

For a smooth $0 \le \chi \le 1$ equal to one on $[1, \infty)$ and zero on $(-\infty, 1/2]$, multiply the packet by $\chi(G/\epsilon)$. The unipotent factors act on the Gram matrix by unitriangular operations and the compact stabilizers fix the specified frames. Consequently G is constant on the orbit, and the smoothing multiplies the orbital operator by exactly the same scalar. This preserves the phases of the Bessel character. For a finite nonempty set of good rational orbits the minimum of G_w over all these orbits and real places w is positive. Below that minimum their joint Gram matrix is unchanged. For an absolutely summable signed row $\sum_p a_p V_p$, first choose a finite set whose omitted sum of $|a_p|\|V_p\|$ is small. The norm error below its cutoff is at most twice that omitted sum. This proves convergence for each such row. Smoothed orbital operators and finite sets

On the regular boxes there is also a rate. In the measure of (5), passing to the leading minors gives density proportional at one place to

$$G^{k-1} \prod_{j=1}^{r} |D_j|^2 \prod_{j=1}^{r+1} 2\, d^2 D_j.$$

The triangular substitution $D_j = c_1 \cdots c_j$ has complex Jacobian $\prod_j D_{j-1}$, with $D_0 = 1$. Fix $|D_j| \le R$ and keep every early minor above $\kappa > 0$ at every place. The orbit majorant of §3 is bounded there for the fixed pair. Holding the early minors fixed, polar coordinates in D_{r+1} give the integral $2\pi\epsilon^k/k$ of $G^{k-1}\, 2d^2 D_{r+1}$ over $0 < G \le \epsilon$. Integrating the bounded early coordinates and taking the union over real places gives an $O_{R,\kappa,u,v}(\epsilon^k)$ bound for the removed squared energy. The constants can depend on the pair and the lower bound for its early minors. Gram-coordinate calculation and regular-box bound

The next step sums a positive packet test on a coset of its period lattice. Under the fitted Fourier and convergence estimates, Poisson summation bounds that sum by the zero Fourier coefficient plus the absolute mass of the nonzero dual coefficients, divided by the lattice covolume. The zero coefficient is the integral controlled at the preceding level. The estimate on the nonzero coefficients must be uniform in the smoothing parameter for the permitted derivative words. Pointwise convergence of the finite Gram matrix above supplies the final passage back to those finite sets. The full word-pair estimate still uses uniform Fourier decay, the weighted bound for the rotated rational orbits, and agreement of the finite and infinite-place mass normalizations. Poisson estimate on the specified lattice · Arithmetic and Bessel factors

4. Boundary terms in the rational sum and the orbit count

Write the geometric sum in (2) as $K^{\mathrm{id}} + K'' + K^{\mathrm{gen}} + K^{\mathrm{ng}}$. For a rational term with first flag F and transported flag F', let D_j be the leading minors of their pairing matrix over the CM field. The algebraic Gram invariant is $D_{r+1}\overline{D_{r+1}} - TT' D_r \overline{D_r}$, with the CM involution in the bars. Its zero locus contains the identity terms and the remaining terms summed in K''. Away from that locus, genericity requires an admissible matrix ξ whose prescribed span equals the localized lattice of the pair. At level primes the span uses the columns $g^{-1}(\mathrm{frame}(F') - \mathrm{frame}(F)\xi)$ and the first lifted flag, where g is the level integer.

Away from those primes it is the sum lattice specified by the flags. The second moment compares pairs of terms with the same row label $(\Lambda, \Lambda', \text{residue Levi element})$. Its corank is $s_0 = k - \dim_E \Theta$, where Θ is the first projection of the space of pairs in the two orthogonal complements that agree modulo the first isotropic space and agree after transport modulo the second. In corank one, the scalar test asks that the transport preserve the relations with specified coefficients of both distinguished vectors. Algebraic Gram invariant · Genericity and scalar relation · Corank · Four parts and same-row pairs · Concrete row labels

Release inputs (identity and zero Gram). For distinct words the identity cells are empty and the nonidentity zero-Gram summands vanish term by term under the K-two and concrete ties. The vanishing proof uses the stabilizer-character and compact-type comparison with averaging on a rational cocompact subgroup. In particular a nontrivial unitary character kills a covariant integrable average: translating by an element on which the character has value $\zeta \neq 1$ gives $I = \zeta I$. It is the release's rational and local analysis that supplies the applicable character or type test. For equal words an identity term survives once on each equal-class block. Its domain has μ_{N_u}-mass $\mu_{N_u}(F_{N_u})$, the real character phase cancels the fixed finite phase, and compact Schur averaging gives the scalar $\mu_{H_u}(H_u)/\dim \pi_u$. For a unit vector in an irreducible type, the compact average of its rank-one projection commutes with the representation, and its trace is the compact volume. This proves that scalar with the normalization of the identity packets. Substitution in (4) gives 1, hence $K^{\mathrm{id}} = I$ on the diagonal. The surviving same-word nonidentity zero-Gram terms have index $1 \le j \le r$, where $j = r - \dim_E(X_F \cap X_{F'})$ for the two isotropic spaces. Relevance compares traces of the two infinitesimal characters on the common nilpotent Lie algebra. The real Bessel estimate and finite type and coset counts bound the squared norm by $2d_u(\sum_{j=1}^{r} H_j)^2$ with $H_j = e^{Cm^6 f \log b}(C/k)^{j(k-1-j)f}$ for a source constant C. The large-weight hypothesis $C'm^6 \log b \le k \log k$ with the specified threshold C' bounds their total by $\|K''\|_{\mathrm{HS}}^2 \le d_u e^{-kf \log k}$, where $f = [F : \mathbb{Q}]$ and the rank and large-k hypotheses are those of the selected family. Their estimate uses (3) on the rational stabilizer quotient. Distinct-word vanishing · Identity cells and phases · Type and boundary ranks · Rank index · Boundary amplitude pairs · Numerical boundary estimate

The rational zero-Gram contribution for distinct words

Fix an ordinary selected family point and use the concrete adelic setting of §2, with its K_2 level, prescribed real packets and common row space. Let E/F be the host CM extension over its maximal totally real subfield, $g = g_1 g_2$ the product of the two distinct rational level primes, and $f = [F : \mathbb{Q}]$. Write $\mathrm{e}(x) = \exp(2\pi i x)$. For words $u \neq v$, every rational term t of (2) with algebraic Gram invariant $G_t = 0$ has zero normalized block. In particular $K_{uv}^{\mathrm{id}} = K_{uv}'' = 0$. Here the identity condition on a rational class implies $G_t = 0$. Indeed in an identity representative the transported isotropic vectors equal the first ones modulo the preceding flag steps, and the two distinguished vectors agree modulo the isotropic space. For positive flag length the first r columns of the cross-pairing matrix are zero. For $r = 0$ the distinguished vectors agree, and their common norm gives $G_t = 0$ with $D_0 = 1$. The invariance of G_t under the rational term equivalence is part of the release's term dictionary. Identity representatives Identity implies zero Gram Representative invariants Selected off-diagonal assembly

The primitive character at a level prime. The finite additive character $\psi_{E,f}$ in the concrete setting is trivial on the integral finite adeles. On diagonally embedded $a \in E$ it has value $\mathrm{e}(-\operatorname{Tr}_{E/\mathbb{Q}} a)$, since the real factor is the product over the f conjugate pairs of $\mathrm{e}(\sigma(a) + \overline{\sigma(a)})$. Each of g_1, g_2 splits completely in E. For $p \in \{g_1, g_2\}$, let $h = g/p$ and choose a prime $\mathfrak{P} \mid p$ in E. Choose $\epsilon_{\mathfrak{P}} \in \mathcal{O}_E$ with residue 1 at $\mathfrak{P}$

and 0 at all other primes dividing $g\mathcal{O}_E$. These prime ideals are pairwise coprime and their product is $g\mathcal{O}_E$. The trace on $\mathcal{O}_E/p\mathcal{O}_E \simeq \prod_{\mathfrak{Q}|p} \mathbb{F}_p$ is the sum of the coordinates, as one sees by reducing the matrix of multiplication on $\mathcal{O}_E$ modulo p. Hence $\mathrm{Tr}_{E/\mathbb{Q}}\, \epsilon_{\mathfrak{P}} \equiv 1 \pmod{p}$. The lift lies in $h\mathcal{O}_E$, so $\mathrm{Tr}_{E/\mathbb{Q}}\, \epsilon_{\mathfrak{P}} = hb$ with $b \equiv h^{-1} \pmod{p}$. It follows that the release's residue character $\psi_{\mathfrak{P}}$ has the exact value

$$\psi_{\mathfrak{P}}(\bar{m}) = \psi_{E,f}(\epsilon_{\mathfrak{P}} m/g) = \mathrm{e}(-bm/p), \qquad b \equiv h^{-1} \pmod{p}.$$

A change of lift changes the argument by an integral adele. The displayed character has kernel $\{0\}$ on $(\mathbb{F}_p, +)$. In particular its restriction to the selected factor over each place of F above p has conductor exponent one after scaling by g^{-1}. For an integral finite adele C, lift each residue at the primes over g by the corresponding $\epsilon_{\mathfrak{P}}$. This gives $\psi_{E,f}(C/g) = \prod_{\mathfrak{P}|g} \psi_{\mathfrak{P}}(C \bmod \mathfrak{P})$. At a conjugate pair the two characters have the same coefficient $b \bmod p$, so their product on residues m_1, m_2 is $\psi_{\mathfrak{P}}(m_1 + m_2)$. This proves the primitivity and the combination of the two residues used in the finite K_2 character. The integral frame comparison identifying that residue sum for χ_2 with the entry sum ϕ_J in the conjugated finite matrix coordinates is a **release input**. Adelic character and sign Complete splitting Chinese remainder lifts Residue character Character on both residues Finite frame comparison

Comparison on the rational stabilizer. Write N_Λ for the rational isometries preserving Λ which lower the isotropic flag by one step under $n-1$ and satisfy $(n-1)X^\perp \subset X$, where X is its top isotropic space. For a term $t = (\Lambda, \Lambda', [\gamma])$, put

$$R_t = N_{u,\infty} \cap \gamma_\infty N_{v,\infty} \gamma_\infty^{-1}, \qquad \Gamma_t = (N_\Lambda \cap \gamma N_{\Lambda'} \gamma^{-1})_\infty.$$

The release supplies $\Gamma_t \subset R_t$, the discreteness of Γ_t, its cocompactness in R_t, and integrability of the packet product on the chosen fundamental domain Ω_t. Groups and domain Discreteness Cocompactness Convergence in the concrete setting

Let K_N be the subgroup of integral finite isometries which satisfy the first two congruences of K_2, on the reference isotropic basis e_a^0 and distinguished vector z^0. Use indices $0 \le a < r$, and set $q_a^0 = e_{a+1}^0$ for $a < r-1$, $q_{r-1}^0 = z^0$. For the duals f_a^0 write

$$B_N(k) = \sum_a H(f_a^0, k q_a^0), \qquad \chi_N(k) = \psi_{E,f}(B_N(k)/g).$$

Here H is the finite Hermitian form, conjugate-linear in its first variable. Modulo g, the congruences make $k-1$ lower each stage of the reference flag, including the stage obtained by adding z^0, by one. In particular $(k-1)(k'-1)q_a^0$ is in the span of the e_b^0 with $b < a$ modulo g. The congruence dual f_a^0 annihilates this span and pairs to zero with q_a^0 modulo g. Thus

$$B_N(kk') - B_N(k) - B_N(k') = \sum_a H(f_a^0, (k-1)(k'-1)q_a^0) - B_N(1) \equiv 0 \pmod{g}.$$

The reference pairing is integral on these lattice vectors, so the congruence implies $\chi_N(kk') = \chi_N(k)\chi_N(k')$. This argument uses the nilradical part of the finite character. The full χ_2 includes the additional entries from the compact-complement frame and is evaluated on K_2 in the block formula. Root coordinates and level Nilradical character Congruence subgroup and multiplicativity

Choose the release's base column classes of Λ, Λ', with finite representatives $A = h_\Lambda x_u$, $B = h_{\Lambda'} x_v$. Two integral comparisons are used here as **release inputs**. The lattice dictionary and the lift of the flags give $\kappa = A^{-1}\gamma_f B \in K_N$. For $n \in N_\Lambda$ the nilradical comparison gives $A^{-1} n_f A \in K_2$ and

$\chi_N(A^{-1}n_fA)=\psi_{u,f}(n_f)$, and likewise on the other side. For n in the common rational stabilizer set $n'=\gamma^{-1}n\gamma$. Then

$$A^{-1}n_fA=\kappa(B^{-1}n'_fB)\kappa^{-1}.$$

By multiplicativity on K_N the two finite characters of n_f, n'_f agree. On each rational unipotent the product of the finite and infinite characters is 1, since both evaluate the same rational root sum divided by g. Consequently the relative character

$$\xi_t(s)=\psi_{u,\infty}(s)\,\overline{\psi_{v,\infty}(\gamma_\infty^{-1}s\gamma_\infty)}\qquad(s\in R_t)$$

is trivial on Γ_t. The use of K_N allows the base transporter κ to have only the first two level congruences. Base lattice and lifted-flag congruences Nilradical comparison Finite comparison on the common stabilizer Product formula for rational unipotents

The rational trace test. The integrand $\mathcal{W}_t(x)=W_u(x)\otimes W_v(\gamma_\infty^{-1}x)^*$ satisfies $\mathcal{W}_t(sx)=\xi_t(s)\mathcal{W}_t(x)$ on R_t. By the finite comparison it descends to $\Gamma_t\backslash G_\infty$. For the inherited Haar measure there, disintegrate along $\Gamma_t\backslash R_t$ over $R_t\backslash G_\infty$. Both G_∞ and the real unipotent group R_t are unimodular. The measures in this disintegration are taken compatibly with the ambient Haar and counting measure on Γ_t. If $\xi_t(s_0)\neq 1$ for some $s_0\in R_t$, right translation by s_0 multiplies the character average on $\Gamma_t\backslash R_t$ by $\xi_t(s_0)$ and preserves its finite measure. Hence the fibre integral and O_t vanish. Group and character covariance Averaging on the common unipotent group

For a pair of rational flags $F_u, \gamma F_v$, let $\ell_{F_u}(Y)$ denote the sum of the simple-root coefficients of Y on the first flag. The coefficients are those of e_a in Yq_a modulo the span of the preceding e_b, with $q_a=e_{a+1}$ or z_u at the last step. The rational relevance test compares $\mathrm{Tr}_{E/F}\,\ell_{F_u}(Y)$ with $\mathrm{Tr}_{E/F}\,\ell_{\gamma F_v}(Y)$ for all Y in the common rational nilpotent Lie algebra. If t is not relevant, there is such a Y with

$$\delta=\mathrm{Tr}_{E/F}(\ell_{F_u}(Y)-\ell_{\gamma F_v}(Y))\neq 0.$$

Take $a=g/(2f\delta)\in F$ and $s_0=\exp(aY)_\infty\in R_t$. The exponential is a finite polynomial and is unitary because aY is skew Hermitian. Powers Y^jq_b for $j\geq 2$ land before e_b by flag lowering. Thus the simple-root sums of the exponential on the two flags are $a\ell_{F_u}(Y)$ and $a\ell_{\gamma F_v}(Y)$. The resulting quotient is

$$\xi_t(s_0)=\mathrm{e}\big(g^{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(a\delta)\big)=\mathrm{e}(1/2)=-1.$$

In this construction s_0 is required to belong to the real common unipotent group. The finite congruences control the lattice Γ_t on which the quotient character is trivial. Relevance and representative Nilpotent exponential and root coefficients Rational witness and its phase

Relevant zero-Gram terms. Suppose t is relevant and $G_t=0$, and consider the case $\xi_t=1$ on R_t, since the preceding averaging applies whenever it is nontrivial. The **release input** of the relevant zero-Gram normal form and its compact lift has the following consequence. If S is the sum of the two rational frame spans, its real orthogonal complement at each real place w contains a positive k-plane L_w. Acting by $M\in U(L_w)$ there and by the identity on $L_w^\perp$ and at all other places fixes both frames pointwise. In the two packet types this acts at w by M and $C_wMC_w^{-1}$, respectively, in unitary coordinates, for a fixed $C_w\in U(k)$. The normal form obtains L_w by correcting an orthonormal frame of the first positive complement by isotropic vectors. On the second side the corrected frame differs from a unitary change of

this frame by vectors in the common isotropic space, which the lifted group fixes. Rational normal form and base change Compact lift and action

Let H_w be this lifted compact group at one real place. It normalizes R_t because it fixes both frames. Its action by conjugation on a Haar measure of R_t has modulus one, since the modulus is a continuous homomorphism from H_w to $\mathbb{R}_{>0}$. The packet product is now R_t-invariant, so O_t is the integral on $R_t\backslash G_\infty$ multiplied by $\mathrm{vol}(\Gamma_t\backslash R_t)$ in the above disintegration. The map $R_t x \mapsto R_t h x$ for $h \in H_w$ preserves the quotient measure by the Haar formula. Both Bessel characters of h are 1 because h fixes both ordered frames. It follows by changing variables that

$$\pi_u(h) O_t = O_t\, \pi_v(\gamma_\infty^{-1} h \gamma_\infty).$$

This change of variables uses the integral on $R_t\backslash G_\infty$ and requires no invariance of the lattice Γ_t under h. Pointwise stabilizer and its character Release covariance

In the selected family the words inject into their Boolean letter strings. A differing letter at index j changes entry $j+1$ of the compact $U(k)$ parameter at every place. The entry is the common parameter at j minus $1/2$ and minus Δ times the Boolean letter, where $\Delta > 0$ is the selected gap. The two irreducible local types are therefore inequivalent. Contract O_t against arbitrary pure tensors in the factors away from w on both sides. This gives an intertwiner of the two local types, one composed with the inner conjugation by C_w. Inner conjugation preserves the isomorphism class, so Schur's lemma kills this intertwiner. Since pure tensors span, $O_t = 0$. Letter formula Word separation at every place Compact-type comparison

Finally, each matching-class block of (4) has the multiplier $\mathrm{vol}(K_2)\chi_2(k(\rho,\rho',t))/\sqrt{c_u^X c_v^X}$ times O_t, with $c_u^X = \mathrm{vol}(K_2)\mu_{N_u}(F_{N_u})\mu_{H_u}(H_u)/\dim \pi_u$ and the corresponding c_v^X. Each unmatched block is zero by definition. The normalized contribution is thus identically zero, term by term, including any identity blocks. Block normalization Zero-term assembly

The rational nonzero-Gram contribution for distinct words

Fix two distinct words u, v at the same ordinary selected point (x, i). Use the real scale 1, $m = 2r+2$, and the same measures and split comparison primes as in the zero-Gram argument. Set

$$D_{uv} = d_u d_v / D_\star, \qquad D_\star = d_\sigma\, \mathrm{volG}\,/\mu_{K''}, \qquad q_{uv} = d_\sigma I_{\mathrm{oo}},$$
$$\varepsilon_{\mathrm{loop}} = \exp(-kf\log k/3), \quad \varepsilon_{\mathrm{ng}} = 2^{-(k-2r-5)}\exp(C_n m^5 f \log b), \quad P_g = (1+3/g_1)^f (1+3/g_2)^f.$$

Here $\mathrm{volG} = \mathrm{vol}(G(F)\backslash G(\mathbb{A}_F))$ and $\mu_{K''}$ is the finite level volume in the tied mass convention. Proposition 3 gives $0 \le q_{uv} \le 1$. The comparison constants and the choice of setting precede u, v. The source's strict family and selection thresholds apply at every eligible index i of the ordinary family. Selected statement · Ordinary selected proof · Size and loop hypotheses in the selection

Weighted same-row expansion

For a rational term t, write $\lambda(t) = (\Lambda, \Lambda')$ for its two lattices and $i(t)$ for the residue Levi element of the base transporter κ_t. The row is $(\lambda(t), i(t))$. The vector v_t in the geometric Hilbert space is supported at $\lambda(t)$, with value

$$\sqrt{w_u(\Lambda)w_v(\Lambda')}\,\frac{\operatorname{vol} K_2}{\sqrt{c_u^X c_v^X}}\,\chi_N(\kappa_t)O_t, \qquad w_u(\Lambda)=|\operatorname{Aut}_{\mathrm{ind}}(\Lambda,F_u)|^{-1}.$$

Here $\operatorname{Aut}_{\mathrm{ind}}$ is the induced automorphism group on the orthogonal quotient of the top isotropic space, O_t is the real integral over the rational intersection quotient, and χ_N is the nilradical character of the preceding zero-Gram section. The full χ_2 of a column block supplies the remaining Levi matrix. In particular a pair in one row has the factor $w_u(\Lambda)w_v(\Lambda')$, and the phase in $\langle v_t, v_{t'}\rangle$ is $\overline{\chi_N(\kappa_t)}\chi_N(\kappa_{t'})$ with the first slot conjugate-linear. Induced group and row weight · Vectors and residue Gram matrix · Block expansion

At a split residue place of size q, let $U\subset \mathrm{GL}_k(\mathbb{F}_q)$ be upper unitriangular, ψ_U the primitive superdiagonal character extended by zero, and $\gamma_q(a)=|U|^{-1}\sum_y \psi_U(yay^{-1})$, summing over $\mathrm{GL}_k(\mathbb{F}_q)$. Then $\gamma_q(1)=|\mathrm{GL}_k|/|U|$. For a unitary irreducible representation π, the operator $|U|^{-1}\sum_{a\in U}\psi_U(a)\pi(a)$ is the orthogonal projection onto the ψ_U^{-1}-space. Summing its conjugates and taking traces gives the eigenvalue of convolution by $\gamma_q/\gamma_q(1)$, namely $|U|\dim(\pi^{U,\psi_U^{-1}})/\dim\pi$. Complex conjugation gives the same bound for the matrix convention of the release. The finite Gelfand–Graev multiplicity-one and generic degree bounds give

$$0\le \frac{|U|\dim(\pi^{U,\psi_U^{-1}})}{\dim\pi}\le \prod_{j=1}^{k-1}(1-q^{-j})^{-1}\le 1+3/q.$$

The degree bound arises by restricting to the last-column abelian radical of the mirabolic. Its $q^{k-1}-1$ nontrivial character spaces have equal dimension by the transitive GL_{k-1}-action, and the chosen character space carries the lower-rank Whittaker character under its stabilizer. Induction gives $\prod_{j=1}^{k-1}(q^j-1)$. For the last numerical inequality, write $y=1/q$. The product for $j\ge 2$ is at least $1-y^2/(1-y)$, before taking reciprocals, so the full inverse product is at most $(1-y-y^2)^{-1}\le 1+3y$ for $0\le y\le 1/3$. Taking the product over the f places above each level prime gives $0\preceq G_{\mathrm{Levi}}\preceq P_g\Gamma_1 I$, where $\Gamma_1=\prod_{\mathfrak{p}\mid g}\gamma_{q_\mathfrak{p}}(1)$. Finite convolution normalization and degree estimate · Finite product matrix

For a set S of terms put $v_i(S)=\sum_{t\in S:i(t)=i}v_t$. Expand the finite matrix inequality against these vectors. Orthogonality between different lattice supports and absolute convergence yield

$$\|K(S)\|_{\mathrm{HS}}^2\le P_g\Gamma_1\sum_i\|v_i(S)\|^2=P_g\Gamma_1\sum_{\substack{t,t'\in S\\ \mathrm{row}(t)=\mathrm{row}(t')}}\Re\langle v_t,v_{t'}\rangle. \tag{7a}$$

Reversing the order conjugates the summand, so a pair of distinct terms contributes twice its real part. The real parts can have either sign. We take absolute values for the loop classes below, and use the signed expansion in (7a) for the scalar and nongeneric classes. For nonzero-Gram terms of one row the corank s_0 is at least one. For two generic terms split the pairs into $s_0\ge 2$, $s_0=1$ nonscalar, and $s_0=1$ scalar. For two nongeneric terms split at $s_0=1$. Cross terms between the generic and nongeneric sums are bounded by the Hilbert-space triangle inequality. Pair definitions and convergence · Gram and pair split · Abstract pair split

Loop counts

Let $X=kf\log k$ and let $\Gamma=\exp(C_{\mathrm{Gam}}m^5 f\log b)$ be the column-window parameter, distinct from Γ_1. The ordinary window is $C_0^{-1}\le d_w\Gamma^2/D_\star\le C_0$ for every word w, with $C_0=3$ in the selected assembly.

In particular $1 \le C_0 \Gamma^2 d_v / D_\star$. The loop estimate begins with $|\langle v_t, v_{t'} \rangle| \le \|v_t\| \|v_{t'}\|$. In the passage from concrete to lattice pairs, the sum of second-lattice weights in a fibre is at most one. The first-lattice weight remains. Summing it gives the genus mass of the first rows, which, multiplied by $\Gamma_1 \dim \pi_u$, is d_u by the tied volume identity. These are weighted counting statements. Replacing the weights by unit multiplicities would change the mass. Weighted fibre inequality · First-row mass and use of the window

For each stratum the count bounds the sum of $w_u(\Lambda) A_t A_{t'}$ over any finite set of lattice pairs, where A_t is the norm of the normalized orbital operator before the row weights. Its right-hand side is the first-row genus mass times $\dim \pi_u$ times one of the factors

$$\frac{d_v}{D_\star} B_s, \quad D_s, \quad \frac{d_v}{D_\star} B_1^*$$

for the nondegenerate and degenerate loop strata of corank $2 \le s \le k$, and the generic nonscalar corank-one stratum, respectively. Here degeneracy refers to the loop stratum inside the nonzero-Gram set. For a fixed sufficiently large C and a nonnegative count factor Ξ_s, the bounds are

$$\begin{aligned} H_0 &= b^{Cm^2 f}, \\ R_s &= (k v_{\mathrm{host}})^{kf} \Xi_s C^{kf} H_0^{C(s+m)m^2} (Csf)^{6sf} \Gamma, \\ B_s &= (2/g)^{(s-1)kf} R_s, \\ D_s &= (2/g)^{(s-2)kf} R_s \, e^{Cm^6 f \log b} (C/k)^{kf}, \\ B_1^* &= C^f H_0^{Cm^3} \Gamma e^{Cm^6 f \log b} \big(C g^{-(2r+1)} k^{-r} \big)^{kf/2}. \end{aligned} \tag{7b}$$

The v_{host} in $k v_{\mathrm{host}}$ denotes the host covolume parameter. The loop estimates use the weighted mean count of primitive sublattices over a genus, the Hermitian determinant and extension-class bounds, and partner counts. The archimedean factors use the large-gap coefficient bounds and the stabilizer covolume estimate on relevant terms. These inputs must apply to the same word-pair lattices and quotient measures. Weighted pair and strata · Three stratum bounds · Genus count and partner inputs · Amplitude and covolume inputs

The level saving in B_s is visible before summing over Gram data. In a nondegenerate bin with Gram parameters $G_1, G_2 > 0$, norm parameter N_T and remaining count factor ΞW, the count is bounded by

$$(2/g)^{(s-1)kf} \big(\sqrt{G_1 G_2} / N_T \big)^k \, \Xi W.$$

The two squared-amplitude estimates bound $E (G_j / N_T)^k A_j^2$ by $\ell_j \tau$, with $E = (e\pi / (k v_{\mathrm{host}}))^{kf}$ and $\ell_j, \tau \ge 0$ from the real bound. Multiplying the count by $A_1 A_2$ and taking the geometric mean of the two estimates cancels $(\sqrt{G_1 G_2}/N_T)^k$, giving the factor $(2/g)^{(s-1)kf} E^{-1} \Xi W \sqrt{\ell_1 \ell_2} \tau$. The degenerate bin has one fewer level direction. Its count also has factors $T^{-f} K_1 K_2$ inside the k-th power. Here $T, K_1, K_2 > 0$ are the bin parameters in the source's degenerate bound. They cancel against the geometric mean of the factors $K_j^{2k} T^{-kf}$ in the two squared-amplitude estimates. In particular the level factor in D_2 is one, and that stratum inherits the k^{-kf} real saving. Count and amplitude cancellation

In the nonscalar corank-one bound, after the congruence comparison the two real parameters at each place are written as $h_{j,w} = \sinh u_{j,w} > 0$, $c_{j,w} = \cosh u_{j,w}$ for $j = 1, 2$. The product lower bound is $(\sqrt{T T'}/g)^f \le \prod_w (c_{1,w} + c_{2,w}) / (h_{1,w} h_{2,w})$. Since $c_{j,w} \le 1 + h_{j,w}$, each factor is at most $4 / (\min(1, h_{1,w}) \min(1, h_{2,w}))$. Consequently the product of the two truncated products $P_j = \prod_w \min(T' h_{j,w}^2, T')$ is at most $(4g \sqrt{T'/T})^{2f}$, and at least one P_j is at most $(4g \sqrt{T'/T})^f$.

Combining this with the real amplitude bound on that member and taking the geometric mean of the two available product bounds is the source of the square root in B_1^*. Nonscalar small-cell bound · Small-cell factor in the amplitude

More precisely, with N_T the positive norm parameter used in the loop count, the source requires $4 \le m \le f \le k, 2 \le b, g \ge (kb)^8, 1 \le v_{\mathrm{host}} \le b, 1 \le N_T \le b^{Cmf}, 1 \le \Gamma \le e^{Cm^5 f \log b}$, and $k \ge K_0 m^5 f \log b$, where K_0 depends on C. The two Ξ bounds are $(kv_{\mathrm{host}})^{kf}\Xi_s \le H_0^C C^{skf}$ for $2 \le s \le k-2$, and $\Xi_s \le (Ck)^{kf}(H_0^{Cm}N_T)^{Ck}$ for $s = k-1, k$. Let $R_s^+ = R_s\Gamma^2 e^{Cm^6 f \log b} C^{kf}$. Terms with $\Xi_s = 0$ vanish. In the middle range,

$$\frac{\log R_s^+}{kf} \ll_C s + 1 + \frac{(s+m)m^4 \log b}{k} + \frac{s\log(Ck^2)}{k} + \frac{m^6 \log b}{k} \ll_C s + 1.$$

Here $m^6 \log b/k \le m/(K_0 f) \le 1/K_0$. Increasing K_0 gives $\log R_s^+/(kf) \le (s - 2 + 1/4)\log k$ throughout the middle range, including $s = 2$. In the top two rows the bound is $O_C(\log k + (1+m^4)\log b)$, using $s \le k, m \le k$ and the top Ξ bound. This is at most $\frac{1}{4}(s-2)\log g$ for K_0 sufficiently large. With $\log 2, \log k \le \frac{1}{8}\log g$, these bounds give each $B_s \le g^{-kf/2}e^{-X}$ and each $\Gamma^2 D_s \le e^{-3X/4}$. In B_1^*, the prefactor in (7b) has logarithm $O_C(kf)$ by $k \ge K_0 m^5 f \log b$ and $m \le f$. The $g^{-(2r+1)kf/2}$ factor absorbs this prefactor while giving at least e^{-X}. Summing the bounds yields

$$\sum_{s=2}^{k} B_s \le g^{-kf/2}, \qquad \Gamma^2 \sum_{s=2}^{k} D_s \le e^{-X/2}, \qquad B_1^* \le e^{-X}.$$

The first term is at most e^{-4X}. Multiplying by the fibre and genus bounds, and using the lower column window on the degenerate stratum, bounds the absolute pair sum of the union by

$$A_{\mathrm{lp}} D_{uv}(e^{-4X} + e^{-X/2} + e^{-X}) \le e^{-X/3} D_{uv} \quad \text{if } 6\log(3A_{\mathrm{lp}}) \le X,\ A_{\mathrm{lp}} \ge 1.$$

Indeed $3A_{\mathrm{lp}} e^{-X/2} \le e^{-X/3}$. The selected route can take $A_{\mathrm{lp}} = \max(1, C_0)$ in the tied row bounds. The union consists of all corank-≥ 2 pairs in the nonzero-Gram set and the generic nonscalar corank-one pairs. In particular $\varepsilon_{\mathrm{loop}} D_{uv}$ bounds each of the two absolute row sums used in (9). Sizes and three exponent bounds · Proofs of the exponent bounds · Absolute sum over the union · Selected loop construction

Nongeneric corank-one pairs

The nongeneric corank-one relation, including equality of rows, is an equivalence relation on the nongeneric terms. Write c for one class. In each class group terms by the rational point $p = (\mathbf{s}, \mathbf{z})$, a composition $\mathbf{s}$ of $r+1$ and its nonzero block parameters $\mathbf{z}$, and the normal label ν. Let $U_{c,p,\nu} = \sum v_t$ over this group. The exact contribution divided by Γ_1 is $\sum_c \| \sum_{p,\nu} U_{c,p,\nu}\|^2$. In particular the two signs of interference in $\Re\langle v_t, v_{t'}\rangle$ remain in the group sums. The release compares

$$\|U_{c,p,\nu}\|^2 \le a^2(c,p,\nu) J_{\mathrm{loc}}(p), \qquad J_{\mathrm{loc}}(p) \le c_{\mathrm{St}} J_B(p).$$

The local scalar factorization uses a common orbital packet in the class at (p,ν) and real quotient measures with the specified stabilizer comparison. Write $\mathrm{rd} = 2v_{\mathrm{host}}, TT' = T_u T_v, P_c = \mathrm{rd}^{12r(r+1)f}(TT')^{6f}$. At fixed c, p, at most $P_c N_G(p)^6$ normal labels contribute, where N_G is the product of real absolute values of the scaled Gram invariant. Let N_D be the product of squared scaled block-end minors, $n(p)$ the real block density and $H(p)$ the product of the finite heights. On the contributing points $N_G, N_D, H \ge 1$ and

$H \le g^f N_G N_D$. The density $n(p)$ includes N_G^{k-1} and the block-parameter powers for the composition, including larger blocks. Class sums and scalar, count and height inputs · Scaled Gram and block density · Concrete finite and real ties

At fixed p unfold the finite sum over (c,ν) to local tuples $d=(d_\mathfrak{p})$, with local weights $w_\mathfrak{p}(d_\mathfrak{p})\ge 0$. For each tuple the sum of class weights in its fibre is at most $V_0\prod_\mathfrak{p} w_\mathfrak{p}(d_\mathfrak{p})$. If Ω is the finite set of places allowing a nongeneric term, the complex finite amplitude is

$$a_{\mathrm{ng}}(d)=\prod_{\mathfrak{p}}(a_{Z,\mathfrak{p}}+a_{N,\mathfrak{p}})-\prod_{\mathfrak{p}}a_{Z,\mathfrak{p}}=\sum_{\varnothing\neq A\subseteq\Omega}\prod_{\mathfrak{p}\in A}a_{N,\mathfrak{p}}\prod_{\mathfrak{p}\notin A}a_{Z,\mathfrak{p}},$$

where $a_N=0$ off Ω. The squared finite amplitude before unfolding is the class weight times $|a_{\mathrm{ng}}(d)|^2$. The weighted second moments of a_Z and a_N are bounded by $H_\mathfrak{p}^{C_1(r+1)^2}n_\mathfrak{p}$ and $q_\mathfrak{p}^{-\kappa}H_\mathfrak{p}^{C_1(r+1)^2}n_\mathfrak{p}$, respectively, with $\kappa=k-2r-5$. Off Ω, $n_\mathfrak{p}=n_{\mathrm{gen},\mathfrak{p}}$. On Ω, the bound is $n_\mathfrak{p}\le 8H_\mathfrak{p}^{C_0'(r+1)^2+2m}n_{\mathrm{gen},\mathfrak{p}}$, with $H_\mathfrak{p}\ge q_\mathfrak{p}\ge 2$. Here C_0' denotes the local mass exponent, independent of the column-window constant. These local inequalities and weighted unfolding are quantitative inputs. Local amplitudes, weights and moment bounds · Pinned local estimate in the family · Selected nongeneric comparison

For each nonempty A, the product of local squared norms has a factor $\prod_{\mathfrak{p}\in A}q_\mathfrak{p}^{-\kappa}\le 2^{-\kappa}$. Apply Cauchy–Schwarz over the subsets and sum in d by factorization. This costs at most $(2^{|\Omega|}-1)^2\le 4^{|\Omega|}$. At each place in Ω, absorb $4\cdot 8\le H_\mathfrak{p}^5$. With $e_7=(C_1+C_0')(r+1)^2+2m+5$ the result is

$$\sum_{c,\nu}a^2(c,p,\nu)\le 2^{-\kappa}H(p)^{e_7}n_{\mathrm{gen}}(p). \tag{7c}$$

This also holds for an empty Ω, when the amplitude vanishes. The appearance of at least one local saving follows from excluding the empty subset before squaring, with the class weights retained. Weighted product argument

For a block composition, the successive end minors z_a^{end} satisfy $z_a^{\mathrm{end}}/z_{a-1}^{\mathrm{end}}=(-1)^{s_a(s_a-1)/2}z_a^{s_a}$, with $z_0^{\mathrm{end}}=1$. They are nonzero algebraic integers on the occupied points. Set $W(p)=\prod_{a,w}(1+|\sigma_w(z_a^{\mathrm{end}})|^2)^2$. We have

$$\sum_{p\text{ occupied}}W(p)^{-1}\le 64^{f(r+1)}. \tag{7d}$$

Indeed distinct algebraic integers of E differ by an element with absolute norm at least one, so their images in $\mathbb{C}^f$ have supremum distance at least one. Take disjoint open products of discs of radius $1/2$ centered at these images. On each disc, $1+|z|^2\le 2(1+|a|^2)$ for center a. Integration of $\prod_w(1+|z_w|^2)^{-2}$ gives at most π^f over the disjoint products, so the sum of the weights of the centers is at most $4^f\pi^f/(\pi/4)^f=16^f$. A composition of length ℓ therefore costs $16^{f\ell}$ for its end minors, and each tuple lifts at most $\prod_a s_a\le 2^{r+1}$ times by the displayed power equations. There are 2^r compositions. This proves (7d) with room in the constant. Block-end formula, support and bound · Pinned counting theorem

Use weighted Cauchy–Schwarz for each class with weight $W(p)P_cN_G(p)^6$ on $U_{c,p,\nu}$. The sum of inverse weights is bounded by (7d) because the number of normals cancels $P_cN_G^6$. Sum the resulting second brackets over the classes and apply (7c). The generic mass comparison is

$$n_{\mathrm{gen}}(p)\le A_{\mathrm{re}}e_1H(p)^{e_N}n(p),\qquad e_N=10(r+1)^2+6m+1.$$

For $A_{\mathrm{exp}} = A(r+1)^2$ with $A \ge C_1 + C_0' + 38$, one has $e_7 + e_N + 6 \le A_{\mathrm{exp}}$. The inequality $H \le g^f N_G N_D$ then absorbs the height and normal factors. The archimedean point estimate used at this stage, for each finite subset of occupied points, is

$$\sum_p W(p) n(p) (N_G(p) N_D(p))^{A_{\mathrm{exp}}} J_B(p) \le \exp(C_B m^5 f \log b)\, I_{\mathrm{re}}.$$

Here I_{re} is the normalized density integral of the nongeneric bound. The selected proof uses the rotated weighted orbital estimate of Theorem B. It lifts each occupied point to a point of the real packet formula, with distinct rational points giving distinct lifts. At a lift, the weight on the left is bounded by the product of the packet weight, its Gram density and the norm weight to the power $A(r+1)^2$, times the weight comparison constant. The squared orbital factor is bounded by the product of the rotated local squares times the Jacobian comparison constant. Positivity and injectivity let one sum over the whole packet point set. Theorem B bounds this weighted sum by $e^{C_B' m^5 f \log b}$ times the packet energy under the uniform gap condition $\mathrm{AllGaps}(\lambda_w, C_g' m^4)$ at each real place. The tied comparison to I_{re} absorbs the weight and Jacobian constants into C_B. This is the archimedean input for the nongeneric count. The resulting majorant from the two brackets is

$$\Gamma_1\, 64^{f(r+1)}\, 2^{-\kappa}\, P_c c_{\mathrm{St}} A_{\mathrm{re}} e_1\, g^{f A_{\mathrm{exp}}} e^{C_B m^5 f \log b} I_{\mathrm{re}}.$$

These bounds hold first on finite sets. Absolute convergence gives the class sums, and the majorants are nonnegative. Mass, point estimate and class bound · Construction of real and finite inputs · Rotated estimate and injective summation · Selected uniform Theorem B call

The exact comparison is $\Gamma_1 A_{\mathrm{re}} I_{\mathrm{re}} \le (\mathrm{rd}/2)^{f(r+1)} g^{f(r+1)(2r+3)} D_{uv}$, with $A_{\mathrm{re}} = \mathrm{rd}^{2r(r+1)(k-1)f}\,\mathrm{AA}$ in the source. Under $0 < \mathrm{rd} \le b$, $1 \le g \le b^{C_g}$, $0 < TT' \le b^{2C_T m}$, $c_{\mathrm{St}} \le b^{C_{\mathrm{St}} m^3 f}$ and $0 \le e_1 \le 2$, the product of the factors apart from $2^{-\kappa} D_{uv}$ has logarithm at most $C_n m^5 f \log b$. This proves the $\varepsilon_{\mathrm{ng}} D_{uv}$ bound for the nongeneric corank-one sum. Frame-passage normalization · Size absorption

The selected finite densities

At a finite place w, put $O = \mathcal{O}_{F_w}$, $A = \mathcal{O}_{E_w}$, $q = |O/\pi O|$ for a uniformizer π, and $a = m = 2r+2$, $M = k + a - 1$. The selected host assumption makes the relative extension split or unramified quadratic at every finite place. In the tied local model write Λ_w for the self-dual A-lattice obtained by base change of the prescribed host lattice L_0. The density test uses primitive a-frames in Λ_w. Primitivity means that the columns form a basis of the intersection of their E_w-span with Λ_w. By Smith normal form it is equivalent to full column rank modulo π, in both components when split. Host and lattice conventions

Write $T = T_u$, $T' = T_v$ and $b_0 = r+1$. Let $D_T = -T e_{b_0} e_{b_0}^*$, with the analogous $D_{T'}$. With Z the rational cross Gram matrix, the rescaled local cross matrix is $C = \Phi_w^* Z_w \Phi_w$, where Φ_w is diagonal with i-th entry $(\delta_w u_P(w))^{-(b_0 - i)}$. Here δ_w is the tied local scalar and $u_P(w) = g_{\mathrm{part}}(w)/(g_1 g_2)$, with $g_{\mathrm{part}}(w) = g_j$ over g_j and 1 elsewhere. Denote the two rescaled frames by x, x'. Off the comparison primes the test uses (x, x'), with Gram blocks $D_T, C, C^*, D_{T'}$. Over either comparison prime g_j the model is split and $\pi = g_j = q$. The norm conditions give $T, T' \in O^\times$ and $T \equiv T' \pmod{\pi^2}$. The frame is $(x, (x' - x)/\pi)$, and its Gram, denoted S_w, is

$$S_w = \begin{pmatrix} D_T & (C - D_T)/\pi \\ (C^* - D_T)/\pi & (D_T + D_{T'} - C - C^*)/\pi^2 \end{pmatrix}. \tag{7g}$$

The same formula follows by conjugating the original block Gram by $B = \left(\begin{smallmatrix} I & -\pi^{-1}I \\ 0 & \pi^{-1}I \end{smallmatrix}\right)$ on the right and B^* on the left. At an occupied point the resulting frame spans the prescribed saturated intersection in a representing row lattice, so its Gram is integral. A nonintegral Gram gives an empty lattice tube. Frame and support in the count · Pinned corank formula

Proved here (primitive density). Put $P_\epsilon(j) = \prod_{i=1}^{j}(1-\epsilon^i q^{-i})$ with $\epsilon = 1$ at split places and -1 at inert places, and $P_\epsilon(0) = 1$. For an integral S_w of reduction corank c_w, the primitive representation count modulo π^j, divided by $q^{j(2Ma-a^2)}$, is

$$\mathrm{densP}_w(Z) = \frac{P_\epsilon(M)}{P_\epsilon(k-1-c_w)} \qquad (j \geq 1). \tag{7h}$$

In the selected range $k \geq a + f + 5$ we have $M \geq a + c_w$. To lift a solution from depth j to $j+1$, the differential is $\overline{X}^* \overline{J}\,\overline{Y} + \overline{Y}^* \overline{J}\,\overline{X}$. Full column rank and self-duality let $\overline{X}^* \overline{J}\,\overline{Y}$ run over all matrices. The map $U \mapsto U + U^*$ onto Hermitian matrices is surjective by choosing the entries above the diagonal and using the residue trace on the diagonal. This works also at residue characteristic two. Each solution has precisely q^{2Ma-a^2} lifts, including when $c_w > 0$.

At a split place write the first Gram component as $\overline{X}_-^{\mathsf{t}} \overline{X}_+$ in dual coordinates. There are $q^{Ma} P_1(M)/P_1(M-a)$ choices for the full-rank $\overline{X}_+$. Extend its columns to a basis. The equation determines the first a columns of the matrix of $\overline{X}_-^{\mathsf{t}}$ in that basis. On the remaining columns only the c_w rows belonging to the left kernel of the prescribed first Gram component must have full rank c_w. This gives $q^{a(M-a)} P_1(M-a)/P_1(M-a-c_w)$ choices. At an inert place split the reduced form into a nondegenerate part of rank $a - c_w$ and its radical. After embedding the nondegenerate part, embed the radical as an ordered isotropic c_w-frame in the complement. Complete this frame to hyperbolic pairs using trace surjectivity. Finite Hermitian spaces of a given dimension are isometric by norm surjectivity, so the unitary group acts transitively on these embeddings. Put $d_0 = M - a - c_w$. The pointwise stabilizer has order $|U_{d_0}(q)| q^{2c_w d_0 + c_w^2}$. In the basis of the isotropic frame, its orthogonal quotient, and the dual frame, the unipotent kernel has one free $d_0 \times c_w$ block D and a $c_w \times c_w$ block L satisfying $L + L^* = -D^* J_0 D$, with J_0 the quotient Gram. This gives the stated order. Dividing $|U_M(q)|$ by it and using $|U_j(q)| = q^{j^2} P_{-1}(j)$ gives the mod-π count in (7h). Finite field count and stabilizer computation

Give Λ_w^a and $\mathrm{Herm}_a(A)$ additive volume one. Then (7h) is the fibre integral of the primitive indicator, since each source residue of depth j has volume q^{-2jMa} and the target tube has volume q^{-ja^2}. For self-dual Haar in the pairings of (7f), a character of conductor ideal $\pi^{t_w} O$ multiplies this integral by $q^{t_w(2Ma-a^2)/2}$. In these bases the trace pairings are perfect at conductor O, and the annihilators for conductor $\pi^{t_w} O$ are π^{t_w} times the respective lattices. For the trace character from $\mathbb{Q}_p$, $\pi^{t_w} O$ is the inverse different of $F_w/\mathbb{Q}_p$. The divided frame Jacobians in source and target are $|N \det B|_w^M$ and $|N \det B|_w^a$, with $|N \det B|_w = q^a$. The fibre integral in the rescaled undivided coordinates therefore has the factor $q^{-a(M-a)}$ against the one in the divided coordinates. The matrix packet has its own additive Haar giving volume one to the order $R_w = \{(z_+, z_-) \in O^2 : z_+ \equiv z_- \pmod{\pi}\}$ at the comparison primes and to A elsewhere. This assigns O^2 volume q at comparison primes. The group level volumes enter the tied mass calculation in (8). Haar and packet normalizations

An independent calculation by halasz_resonator gives the same corank factor and enumerates all three possible coranks for $M = 4,\ a = 2$ over residue fields of sizes two and three. The finite examples also

exhibit the extra singular solutions that result from including nonprimitive frames. Independent proof and finite-field counts

Proved here (reference ratio). For $d_T = \mathbf{1}_{T\in\pi O}$, one word contributes $P_\epsilon(M)/P_\epsilon(k-d_T)$ to the reference product. The factor alpD_w uses both words off the comparison primes and a single factor at those primes, where the norms are units. Thus at occupied points, with $P = P_\epsilon$,

$$R_w(Z) := \frac{\mathrm{densP}_w(Z)}{\mathrm{alpD}_w} = \begin{cases} \dfrac{P(k-d_T)P(k-d_{T'})}{P(M)P(k-1-c_w)}, & w \text{ outside the comparison primes,} \\ \displaystyle\prod_{i=k-c_w}^{k} (1-q^{-i}), & w \text{ at a comparison prime.} \end{cases} \tag{7i}$$

In particular the typed ratio is at most one. Put $n = k - a$. A product over any interval starting at $i_0 \geq 1$ has logarithm between $-4q^{-i_0}$ and $2q^{-i_0}$, by the geometric sum and $-\log(1-x) \leq 2x$ for $0 \leq x \leq 1/2$. The numerator starts at $k - c_w \geq n$, and the reference factors start at k or $k+1$. Since $a \geq 2$, this gives

$$-5q^{-n} \leq \log R_w(Z) \leq 4q^{-n}, \qquad \sum_w q_w^{-n} \leq f \sum_{j\geq 2} j^{-n} \leq 3f\, 2^{-n}.$$

The logarithms of the reference product and of the occupied density products converge absolutely. The log error in the ratio product after keeping the places over rational primes $\ell \leq Y$, for integral $Y \geq 2$, is at most $5fY^{1-n}/(n-1)$. This bound is uniform in the words and occupied Z, including when $c_w > 0$. Write $s_{\mathrm{den}} = \sup_{Z \text{ occupied}} \prod_w R_w(Z)$, with value zero if there are no occupied points. Then $e_1 = \max(1, s_{\mathrm{den}}) \leq \exp(12f\, 2^{-n})$. Reference factors · Finite density and uniform error

Generic scalar corank-one pairs

Proved here (signed finite comparison given the orbital identities). Use the scalar-row coefficients $c_\rho(p)$ and tensor orbital vectors $V(p)$ on the tower of block parameters. At fixed words, the local packet label comes from the integral cross matrix (in the comparison order at a typed place), with the prescribed scalar shift in the marked corner. The marked-corner lift is a **release input**. Given equal kind and equal occupied local labels, it provides integral left and right equivalences of the shifted matrices preserving that corner. They induce an integral invertible basis change on (7g) because the two components of each equivalence agree modulo π. Thus the corank and (7h) are constant on each occupied label. The local coefficients BBtop_w include the prescribed unipotent characters, the phase at a typed final block of length two, and the class isomorphism weights. Marked-corner lift and class congruence · Explicit split basis change and local phases

For a fixed finite point set, let $\lambda = (\kappa, \theta)$ with $\kappa = \mathrm{kind}$ and θ a restricted tuple of local classes. Let b_λ be the product of the BBtop_w for θ, with value zero off kind κ, and let $A_\lambda > 0$ be the product of their class automorphism orders. The **release inputs** here are the signed partner sum giving b_λ in orbit coordinates, the orbital fibre mass identifying the weighted count with the corresponding primitive density product, and constancy of the norm factor on the support of a packet. For a packet of positive norm, the factor in the mass cancels the two square roots in the row Gram matrix. The reference product $\alpha = \prod_w \mathrm{alpD}_w$ also appears under each square root, so the resulting coefficient of $A_\lambda^{-1} b_\lambda b_\lambda^*$ is $\prod_w R_w(Z_\lambda)$ for an occupied representative Z_λ. A packet with zero norm contributes zero to the row kernel. The positive block weight ϕ

contributes a factor ϕ^{-1} on each side of both kernels by the same diagonal congruence. Signed partner and fibre mass identities · Row normalization by the reference product

The top kernel $\mathcal{K}_{r+1}$ is the Gram sum of these packet vectors over the full restricted class set with weights A_λ^{-1}. The source supplies local square-summability and a base class with coefficient and automorphism weight one outside a finite set for each point. On a finite point set take the union of those exceptional sets. Cauchy–Schwarz gives absolute convergence of the pairwise class products at those places. Restricting the top sum to occupied labels removes positive-semidefinite terms. By (7i) each occupied multiplier is nonnegative and at most s_{den}, and restricting the row Gram sum to a finite set of rows R also removes positive-semidefinite terms. Thus

$$\big(\textstyle\sum_{\rho\in R} c_\rho(p)\overline{c_\rho(p')}\big)_{p,p'} \preceq s_{\mathrm{den}}\,\mathcal{K}_{r+1}.$$

The phases stay in the packet vectors throughout the comparison. Testing against the coordinates of the finite family $V(p)$ in an orthonormal basis gives the corresponding bound for the sum of vector squares. The signed off-diagonal main-row identification and the passage to the full scalar-row sum use the tied orbital masses and the real estimates below. The equal-word diagonal bound uses the separate $\mathrm{RowMainEqP}$ comparison. Local class summability and support · Selected row ties · Equal-word scalar comparison · Its place in the family assembly

Here is the quantitative analytic input used with that finite inequality. At a stage i of the tower, group the last s coordinates into a block of length s. The local theta majorant dominates the diagonal block kernel and is constant on the stated residue classes. Its sum over distinct admissible residues is dominated by $(1+e_i)\mathcal{K}_{i-s}$. These are finite quadratic form inequalities, which pass to the adelic product by the Schur product theorem with the ordinary unramified and prescribed level factors. The real Poisson step and packet bounds give the following one-level inequality for a square-root bound σ_i on the smoothed quadratic form, using the notation $\mathrm{vol}(i-s+1,j)$ for the lifting-lattice covolume factors:

$$\sigma_i \;\le\; \sqrt{(1+3^f\epsilon_{\mathrm{lat}})(1+2^f\epsilon_{\mathrm{lat}})(1+e_i)}\,\sigma_{i-1} + \sum_{s=2}^{i}\sqrt{(1+3^f\epsilon_{\mathrm{lat}})(1+e_i)C_4^{fs}\prod_{j=1}^{s-1}\mathrm{vol}(i-s+1,j)}\,\sigma_{i-s}.$$

The inequality refers to the uniform bound for that step over its permitted test words. Here $\epsilon_{\mathrm{lat}} = (C_1'\Theta^{-(K-5/2)})^f$ with $K\ge 4$, and each lifting covolume factor is at most Θ^{-2f}. Divide σ_i by $\prod_{j=1}^{i}(1+e_j)^{1/2}$. The shorter-block coefficients can only decrease after this normalization, since the intervening density factors are at least one. For $\eta = (C_*/\Theta)^f$, take $C_* \ge \max(3C_1', 2C_4, 1)$ and Θ large enough that $\eta \le 1/2$. The length-one coefficient is at most $1+\eta$. A length-s coefficient for $s\ge 2$ is at most η^{s-1}, using $s/2 \le s-1$. The running maximum of the normalized σ_i therefore grows by at most $1+\eta+\eta/(1-\eta) \le 1+3\eta$ at each stage. After squaring, its accumulated factor is at most $\exp(6(r+1)\eta)$, which is $1+O((r+1)\eta)$ on the prescribed smallness range. The source's size estimate and cutoff limit, where the initial normalized square tends to I_N, give

$$\sum_{\rho\in R}\Big\|\sum_{p} c_\rho(p)V(p)\Big\|^2 \le (1+\delta)e_1 I_N, \qquad \delta = C(r+1)(C/\Theta)^f, \qquad e_1 = \max(1, s_{\mathrm{den}}). \tag{7e}$$

In the recursion the lower-level e_i vanish. Scaling the top kernel by e_1 scales its majorants as well and gives just one factor e_1 in (7e). The quantitative input here is the one-level Poisson and block estimate with the summability, lattice sizes and cutoff limit of the selected packets. Local block majorization and Poisson

recurrence · Lifting factors and limit hypotheses · Iteration in the source · Top-only scaling · Application to actual rows

Proved here (nonsingular fibre mean). Write $V = (E^M, J)$ for a nonsingular Hermitian space over the CM extension E/F, and $G = U(V)$. For an ordered a-frame X put $Q(X) = X^*JX$, and for a nonsingular Hermitian matrix T over E/F let $X_T = Q^{-1}(T)$. Use a nontrivial additive character ψ of $\mathbb{A}_F/F$. At each place the measures dx_w, ds_w on frame and Hermitian matrix space are self-dual for the pairings

$$B_X(X,Y) = \operatorname{Tr}_{E_w/F_w} \operatorname{tr}(X^*JY), \qquad B_S(S,B) = \tfrac{1}{2} \operatorname{Tr}_{E_w/F_w} \operatorname{tr}(SB) = \operatorname{tr}(SB).$$

Let $d\nu_{T,w} = |dx_w/ds_w|_{X_T}$ denote disintegration along Q, and $g_w(T) = \int_{X_T(F_w)} \Phi_w \, d\nu_{T,w}$. For $M > 2a$, a smooth compactly supported real test Φ_∞ and locally constant compactly supported finite tests Φ_w equal to the standard lattice indicator almost everywhere, put $\Phi = \Phi_\infty \prod_{w<\infty} \Phi_w$. Then

$$\frac{1}{\operatorname{vol}_{dg}[G]} \int_{[G]} \sum_{X \in X_T(F)} \Phi(g^{-1}X)\, dg = g_\infty(T) \prod_{w<\infty} g_w(T), \qquad [G] = G(F)\backslash G(\mathbb{A}_F). \tag{7f}$$

Here g_∞ integrates the real test against the product of the real fibre measures. This is the nonsingular Gram coefficient of the convergent Siegel–Weil formula in the convention of the pinned statement. Weil treats $M > 2a$ in his 1965 paper. Yamana, pp. 236–237 and Theorem 2, gives the convergence criterion and the factor one for the Eisenstein series of the standard Weil section in this range, with $s_0 = (M-a)/2$. In the unitary case $M - r_W > a$ for the F-Witt index r_W when $M > 2a$. Gan–Qiu–Takeda §3.1 specify the Tamagawa probability measure on the theta side. The proof of (7f) uses the orbit of a single Gram coefficient and its quotient measure. Weil · Yamana · Gan–Qiu–Takeda · Pinned mean and test conventions · Self-dual pairings · Pinned proof of the product normalization

If $X_0 \in X_T(F)$, the ordered span is a nondegenerate copy of the form T. Witt extension makes $X_T(F)$ a single orbit $G(F)/H(F)$ with $H = U(W)$, where $W = \langle X_0 \rangle^\perp$. It has determinant class $[\det J/\det T]$ in $F^\times/N_{E/F}E^\times$, and its real signatures are those of V minus those of T. The orbit fixes the entire ordered frame. If T is represented everywhere locally, choose a global diagonal form of rank $M-a$ with determinant $\det J/\det T$ and the required real signatures. Weak approximation selects the first $M-a-1$ signs, and the determinant specifies the last one. Finite local classification by dimension and determinant and Landherr's local–global theorem identify its orthogonal sum with T with V. Thus in the globally empty case some local fibre is empty, and both sides of (7f) vanish. Local Witt extension gives $G(F_w)/H(F_w)$ at every represented place. At almost every finite place a unimodular frame splits off integrally, and the unimodular complements are isometric by finite-field classification and smooth lifting. The orbit maps therefore give the restricted product quotient. Landherr · Pinned orbit quotient convention

The differential $dQ_X(Y) = X^*JY + Y^*JX$ admits the right inverse $S \mapsto XT^{-1}S/2$ on X_T. It defines the local fibre measures by submersion and makes g_w continuous on the nonsingular locus. Choose rational coordinate volume forms dX, dS and the induced rational fibre form dX/dS by the differential exact sequence. The fibre form is G-invariant since $N_{E/F} \det g = 1$. Choose invariant rational forms on G, H whose quotient on X_T is this form. The self-dual measures are scaled from powers of a self-dual scalar measure on F_w by $|\det B_X|_w^{1/2}$ and $|\det B_S|_w^{1/2}$, using the pairing matrices in the rational coordinates. Their ratio has adelic product one by the product formula. For the trace character the scalar measure gives $\mathcal{O}_{F_w}$ volume $|\mathfrak{d}_{F_w/\mathbb{Q}_p}|_w^{1/2}$ at finite places. Its combined discriminant factor in the quotient is

$|D_F|^{-(2Ma-a^2)/2}$, the Tamagawa discriminant factor for the dimension difference $\dim G - \dim H = 2Ma - a^2$. Replacing $\psi(t)$ by $\psi(bt)$ for $b \in F^\times$ scales local fibres by $|b|_w^{(2Ma-a^2)/2}$, again with product one. The two positive-rank unitary groups have the same determinant torus $U(1)$, so the Tamagawa convergence factors $L_w(1, \chi_{E/F})$ and the global $L(1, \chi_{E/F})$ factor cancel in the quotient. Consequently $dg^{\mathrm{Tam}}/dh^{\mathrm{Tam}}$ on the orbit is the product of the self-dual fibre measures. Kottwitz's formula gives $\tau(U(L)) = 2$ for each positive-rank Hermitian space L, including rank one. The dual centre is $\mathbb{C}^\times$ with quadratic inversion, so its fixed subgroup has order two. A locally trivial cohomology class restricts to a character over E that is trivial by Chebotarev. The resulting class for $\mathrm{Gal}(E/F)$ is zero because the coboundaries for inversion are squares and every complex number has a square root. For $a = 0$ the fibre is a point and (7f) follows directly. Tamagawa measure convention · Kottwitz's formula · Tamagawa numbers

At a finite place, write L_X, L_S for coordinate lattices and q_w for the residue size. The self-dual lattice volume for a pairing with annihilator $L^\perp$ is $([L : L \cap L^\perp]/[L^\perp : L \cap L^\perp])^{1/2}$. A weighted congruence count on $L_X/\varpi^j L_X$ for $Q(X) - T \in \varpi^j L_S$, normalized by $q_w^{j(2Ma-a^2)}$, is converted to $g_w(T)$ by multiplying its limit by $dx_w(L_X)/ds_w(L_S)$. Take L_X to contain the test support and L_S large enough that the congruence is well defined on these residues. Equivalently divide the test integral over the Gram tube by $ds_w(\varpi^j L_S)$ and let j grow. This follows by continuity of the fibre integral and includes ramified or dyadic pairings, nonunit Gram determinants and level weights. At a real place with $\psi_w(t) = e^{2\pi i b_w t}$, the densities against ordinary coordinate Lebesgue measures on tuple entries and on real diagonal and complex upper Gram entries are

$$dx_w = |b_w|^{Ma} 2^{Ma} |\det J_w|^a \, dX_{\mathrm{Leb}}, \qquad ds_w = |b_w|^{a^2/2} 2^{a(a-1)/2} \, dS_{\mathrm{Leb}}.$$

These multiply the coordinate coarea formula by their ratio. The real group chart in (5) uses its separate prescribed quotient measures and $2\, d\Re c\, d\Im c$ on each complex coordinate. Its C_{oo} scales by $A/(BB')$ if the host and the two real stabilizer Haars scale by A, B, B'. Pairing conventions · Pinned finite lattice and volume calculation · Real group disintegration

At a split or unramified finite place with conductor $\mathcal{O}_{F_w}$, unimodular J, T and the standard lattice test, the two additive lattice volumes are one. Finite Witt extension gives the group order ratio of $\mathrm{GL}_M/\mathrm{GL}_{M-a}$ in the split case and U_M/U_{M-a} in the inert case. At inert places the differential on the integral fibre is surjective modulo the prime because the trace of the unramified residue extension is surjective, including in residue characteristic two. At split places the two frames pair a vector space with its dual, and a variation in one frame realizes every target variation. Smooth lifting multiplies the number of solutions by $q_w^{2Ma-a^2}$ per congruence step. Thus the density is

$$g_w(T) = \prod_{i=M-a+1}^{M} (1 - \epsilon_w^i q_w^{-i}), \qquad \epsilon_w = 1 \text{ if split}, \quad \epsilon_w = -1 \text{ if inert}.$$

The tail has summable deviation $O_{M,a}(q_w^{-(M-a+1)})$, hence converges absolutely to a positive product. The finitely many remaining factors are finite by the submersion and the compact test supports. Now unfold the sum over $H(F)\backslash G(F)$ first for $|\Phi|$ by Tonelli. Integration over $H(F)\backslash G(\mathbb{A}_F)$ factors as the covolume of H times the fibre integral on $H(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, with orbit map $g \mapsto g^{-1}X_0$. The preceding local convergence proves integrability and allows the same unfolding for Φ. Dividing by $\tau(G)$ leaves the scalar $\tau(H)/\tau(G) = 2/2 = 1$. With an arbitrary host Haar and the compatible stabilizer Haar, both covolumes scale by the same factor. The finite genus sum induced by this unfolding uses inverse automorphism

weights by orbit–stabilizer. The row weights in (7a) use the induced group on the orthogonal quotient, with the unipotent volume accounted for in the tied normalization. Unramified density formula · Quotient and covolume · Induced row weights

The full lattice indicator used for absolute integration in (7f) majorizes the primitive indicator in the same coordinates. The densities agree for a unit Gram. For a nonsingular Gram with singular reduction, the full density can also have contributions from integral overlattices of the framed span. The selected signed comparison uses (7h) in the prescribed coordinates and the class Gram sum above. Its rational early-minor boundary terms use the block estimates even when the exceptional stratum is Haar-null for (5). Full and primitive lattice densities · Actual scalar comparison

Application to the tied mass. Take $M = 2r + k + 1$ and $a = 2r + 2$ for the joint frame. Its pointwise stabilizer has Hermitian rank $k - 1$. The two single-frame stabilizer comparisons use nonsingular completions of size $2r + 1$. All three sizes are in the range of (7f) under the selected host rank hypothesis. For the given ambient Haar the orbit proof equates its covolume to that of each stabilizer in the Haar compatible with the self-dual fibre measure. The source obtains the same equality by comparing the mean to unfolding on a positive test and cancelling the fibre integral. It then uses the prescribed Haar identifications for the joint frame and both completed single frames to obtain $\mathrm{volB}\,\mathrm{volG} = \mathrm{volHa}\,\mathrm{volHa}'$. The two tied unipotent quotient volumes are one. The Haar identifications and completed-frame construction enter through the selected volume theorem. Compatible covolumes · Positive-test cancellation · Selected frame and Haar transport

The other volume ratios use the actual level indices. At a mass place the ambient natural level has volume $\gamma_{\mathfrak{p}}\mu_{K'',\mathfrak{p}}$, and the two stabilizer levels have volumes $\gamma_{\mathfrak{p}}\mu_{S'',\mathfrak{p}}$ and $\gamma_{\mathfrak{p}}\mu'_{S'',\mathfrak{p}}$, with $\Gamma_1 = \prod_{\mathfrak{p}} \gamma_{\mathfrak{p}}$. Thus the tied local ratio $\mu_{K,\mathfrak{p}}\mu_{K^{\mathrm{nat}},\mathfrak{p}}/(\mu_{S,\mathfrak{p}}\mu'_{S,\mathfrak{p}})$ contributes one reciprocal $\gamma_{\mathfrak{p}}^{-1}$. The ordinary factor in the product uses $\mu_{K,o}^2/(\mu_{S,o}\mu'_{S,o})$ in the source notation. The column mass uses the same stabilizer level, with $c_0 = \mathrm{volN}\,\mathrm{volH}/\dim \pi_u$, $\mathrm{volN}\,\mu_N = 1$ and $d_u = \dim \pi_u\,\mathrm{volHa}/(\mathrm{volH}\mu_{H''})$. This gives $c_0^{-1} = d_u\mu_{S''}/\mathrm{volHa}$ since $\mu_{S''} = \mu_N\mu_{H''}$. The primed column has the corresponding identity. Level, ordinary and column factors · Product of the level factors

To normalize (7e), the coefficient-to-signature comparison contributes $\mathrm{AA}/(c_0c_0')$, where $\mathrm{AA} = \mathrm{volB}\,C_{\mathrm{oo}}(\mathrm{MM}/\mathrm{volZZ})/\mu_K$. The real chart gives $I_{\mathrm{oo}} = C_{\mathrm{oo}}\mathrm{IntC}$. In the natural measure $I_N = (\mathrm{volZZ}/(\mathrm{unitLevelVolume}\,\mathrm{volPP}))(\mathrm{volPP}\,\mathrm{IntC})$, so $I_N = (\mathrm{volZZ}/\mathrm{unitLevelVolume})C_{\mathrm{oo}}^{-1}I_{\mathrm{oo}}$. Multiplying by Γ_1, one uses

$$\frac{\mathrm{MM}}{\mathrm{unitLevelVolume}} = \frac{\mu_{K''}\mu_K}{\Gamma_1\mu_{S''}\mu'_{S''}}, \qquad c_0^{-1} = \frac{d_u\mu_{S''}}{\mathrm{volHa}}, \quad c_0'^{\,-1} = \frac{d_v\mu'_{S''}}{\mathrm{volHa}'}, \quad \mathrm{volB} = \frac{\mathrm{volHa}\,\mathrm{volHa}'}{\mathrm{volG}}.$$

The positive auxiliary mass tt cancels. Before cancelling the remaining volume factors, the scalar real-part sum is at most

$$(1+\delta)e_1\,\Gamma_1(c_0c_0')^{-1}\frac{\mathrm{volB}}{\mu_K}\frac{\mathrm{MM}}{\mathrm{unitLevelVolume}}I_{\mathrm{oo}}.$$

The displayed identities reduce the mass and energy factors to $d_ud_v\mu_{K''}I_{\mathrm{oo}}/\,\mathrm{volG} = q_{uv}D_{uv}$. The generic scalar corank-one real-part sum is therefore at most

$$(1+\delta)e_1\,q_{uv}\,D_{uv}, \qquad q_{uv} = d_\sigma I_{\mathrm{oo}} \in [0,1]. \tag{8}$$

The selected size comparison gives $0 \le \delta \le b^{-7}$. Mass data · Exact mass cancellation · Scalar-row estimate · Scalar comparison and size bounds

Combining the three estimates

Proved here. Write $A = (1+\delta)e_1 q_{uv}$. Apply (7a) to the generic set and to the nongeneric set. The generic squared norm is bounded by $P_g(A + 2\varepsilon_{\mathrm{loop}})D_{uv}$, using the estimate on each of its two loop classes and the scalar estimate (8). The nongeneric squared norm is bounded by $P_g(\varepsilon_{\mathrm{loop}} + \varepsilon_{\mathrm{ng}})D_{uv}$. The two zero-Gram parts vanish for this pair. The Hilbert-space triangle inequality gives

$$\|K_{uv}\|_{\mathrm{HS}}^2 \le P_g \left(\sqrt{A + 2\varepsilon_{\mathrm{loop}}} + \sqrt{\varepsilon_{\mathrm{loop}} + \varepsilon_{\mathrm{ng}}}\right)^2 D_{uv}. \tag{9}$$

The absolute loop estimate bounds the sum over both loop classes together. In (9) each of the two generic terms uses that upper bound, and the nongeneric term uses it for its corank-≥ 2 pairs. The factor q_{uv} enters through (8). Abstract pair bound and size inputs

Put $t = b^{-7}$. The source uses
$b \ge 16,\ k \ge 32,\ f \le k,\ k \ge 4r+12,\ g_j \ge (kb)^8,\ k \ge 70\log b,\ (5/3)f\log k \ge 1$, and $8C_n m^5 f \log b \le k$, together with the density and loop hypotheses above. These give $P_g \le e^{6t}$, $e_1 \le e^{t^2}$, and $\varepsilon_{\mathrm{loop}}, \varepsilon_{\mathrm{ng}} \le e^{-k/5} \le t^2$. For the nongeneric estimate use $k - 2r - 5 \ge k/2$ and $(\log 2)/2 - 1/8 > 1/5$. For the density estimate use $12 \cdot 2^{-(k-2r-2)} \le 2^{-k/2}$ and $f\,2^{-k/2} \le e^{-k/5}$ in the source size range. The latter inequality follows for $k \ge 32$ by $f \le k$ and the decrease of $(\log k)/k$. For $0 \le t \le 1/64$, $e^{6t} \le 1 + 7t$ and $(1+t)e^{t^2} + 2t^2 \le 1 + 2t$, by $e^y \le (1-y)^{-1}$. Thus the final coefficient is at most $(1+7t)(1+5t/2)^2 \le 1 + 13t \le 1 + b^{-6}$. Numerical hypotheses and assembly
Thus under these hypotheses

$$\|K_{uv}\|_{\mathrm{HS}}^2 \le (1 + b^{-6})D_{uv} \qquad (u \ne v). \tag{9a}$$

Uniform selected estimate · Application in the family

For equal words the nonzero-invariant estimate supplied by the diagonal scalar and nongeneric comparisons is $\|K_{uu} - I - K''\|_{\mathrm{HS}}^2 \le \Gamma d_u^2 / D_\star$, with $\Gamma = \exp(C_{\mathrm{Gam}} m^5 f \log b)$. Adding the boundary part and using $\|X+Y\|_{\mathrm{HS}}^2 \le 2\|X\|_{\mathrm{HS}}^2 + 2\|Y\|_{\mathrm{HS}}^2$ yields the diagonal defect constant $e_0 = 2C_0/\Gamma + 2e^{-kf\log k}$ under $d_u\Gamma^2/D_\star \le C_0$. The ambient dimension estimate $D_{\mathrm{all}} \le (1+\eta_1)D_\star$ is the separate row-count theorem. The source's **release input** LinkS says that for a fixed selected family there is x_0 such that for all $x \ge x_0$ and all its indices i, there are η_0, η_1, e_0, with $e_0 \ge 0$ and $\max(\eta_0, \eta_1, \sqrt{e_0}) \le \exp(-x^{\epsilon/8})$, with the following implication. If $C_0^{-1} \le d_u\Gamma^2/D_\star \le C_0$ for every u, then (10) holds for every pair and every word. The chosen proviso ensures the window on the used family. In the off-diagonal construction the setting is chosen for (x, i) before u, v, and the strict family and selection thresholds also apply. Family spectral interface · Diagonal assembly and row count · Selection quantifiers · Eventual quantifier · Chosen proviso

5. Finite-dimensional implications

At a family point in that window, put $a = \sqrt{e_0}$ and $t_{\mathrm{row}} = \dim\ker\Pi$. The three spectral inequalities are

$$\|K_{uv}\|_{\mathrm{HS}}^2 \le (1+\eta_0)d_u d_v/D_\star \quad (u \ne v), \qquad \|K_{uu} - I_{C_u}\|_{\mathrm{HS}}^2 \le a^2 d_u, \qquad D_{\mathrm{all}} \le (1+\eta_1)D_\star \tag{10}$$

We can replace η_0, η_1 by their nonnegative parts. The rank and row inputs are the ramified dimension bounds $r_u \le d_u$, $r_u \ge (1-\Delta_r)d_u$ and $t_{\mathrm{row}} \le \delta_{\mathrm{row}} D_{\mathrm{all}}$ at the selected point. They are **release inputs**, from the selected family and its tied row and column lattices. In the source the per-word discarded proportion is bounded by $2f/b$ under the size and ramified rank hypotheses with $b \ge 3$. The aggregate bound used for the slope has $\sum_u (d_u - r_u) \le d_e \sum_u d_u$. Spectral input · Rank and row hypotheses · Ramified share estimate

Proposition 4 (proved here). For a nonempty set $\mathcal{S} \subseteq \mathcal{U}$ with $M_{\mathcal{S}} = \sum_{u\in\mathcal{S}} d_u > 0$, set $C'_u = A_u V_u$ and $Q_{\mathcal{S}} = \sum_{u\in\mathcal{S}} C'_u (C'_u)^*$. Let $\mathcal{B}_{\mathcal{S}} : \bigoplus_{u\in\mathcal{S}} \mathbb{C}^{r_u} \to H$ be the synthesis map with columns C'_u. Then

$$\begin{aligned} \operatorname{tr} Q_{\mathcal{S}} &\ge (1-\Delta_r-a)_+ M_{\mathcal{S}}, \\ \operatorname{tr} Q_{\mathcal{S}}^2 &\le U_{\mathcal{S}} := (1+a)^2 M_{\mathcal{S}} + (1+\eta_0)\frac{M_{\mathcal{S}}^2 - \sum_{u\in\mathcal{S}} d_u^2}{D_\star}, \\ \|\Pi \mathcal{B}_{\mathcal{S}}\|_{\mathrm{op}} \le \|\mathcal{B}_{\mathcal{S}}\|_{\mathrm{op}} &\le U_{\mathcal{S}}^{1/4}, \qquad \|A_u\|_{\mathrm{op}}^2 \le 1 + a\sqrt{d_u}, \\ \dim_L \sum_{u\in\mathcal{S}} \operatorname{im} B_u &\ge \frac{(1-\Delta_r-a)_+^2 M_{\mathcal{S}}^2}{U_{\mathcal{S}}} - t_{\mathrm{row}}. \end{aligned} \tag{11}$$

Here $y_+ = \max(y, 0)$, and the normalized synthesis bound is for the Euclidean direct sum with its squared norm equal to the sum of source squared norms. Full trace square · Ramified matrix interface

Indeed the trace error from r_u in $\operatorname{tr}(C'_u)^* C'_u$ is $\operatorname{tr}((K_{uu} - I)V_u V_u^*)$, bounded in absolute value by $a\sqrt{d_u r_u}$ by Hilbert–Schmidt Cauchy–Schwarz. Summing proves the first line, including the positive part since the trace is nonnegative. By cyclicity,

$$\operatorname{tr} Q_{\mathcal{S}}^2 = \sum_{u,v\in\mathcal{S}} \|(C'_u)^* C'_v\|_{\mathrm{HS}}^2.$$

Left and right compression by isometries can only decrease each Hilbert–Schmidt norm, as is seen by extending orthonormal bases. On the diagonal apply the triangle inequality to $K_{uu} = I + (K_{uu} - I)$, and off the diagonal use (10) on ordered pairs. This proves the second line. Since $Q_{\mathcal{S}} = \mathcal{B}_{\mathcal{S}} \mathcal{B}_{\mathcal{S}}^*$, its largest eigenvalue is the square of the synthesis norm and is at most $(\operatorname{tr} Q_{\mathcal{S}}^2)^{1/2}$. For a single full block, $\|K_{uu} - I\|_{\mathrm{op}} \le \|K_{uu} - I\|_{\mathrm{HS}}$ gives the separate bound. The identity scale is independent of the source dimensions, but the permitted operator defect grows as $a\sqrt{d_u}$. Matrix identities · Full block normalization

The kernel of $Q_{\mathcal{S}}$ is the intersection of the kernels of $(C'_u)^*$, so its rank equals the dimension of the span of the compressed columns. For its positive eigenvalues λ_j, Cauchy–Schwarz gives $(\sum_j \lambda_j)^2 \le \operatorname{rank}(Q_{\mathcal{S}}) \sum_j \lambda_j^2$. If the trace is zero the lower bound for the rank before row projection is zero. Projection then loses at most t_{row} dimensions. By the Bessel comparison in §1, the single common row change and the wordwise invertible changes preserve the sum of images. A common conjugation also preserves this dimension, and algebraic rank is preserved under extension by the embedding $L \hookrightarrow \mathbb{C}$, by the minor criterion. This proves the last line. Transport of the algebraic blocks · Finite trace comparison

For example, let $M = \sum_u d_u$, assume $D_\star / M \le s$, and let $p = M_{\mathcal{S}}/M$. Equations (10) and (11) imply

$$\frac{\dim_L \sum_{u\in\mathcal{S}} \operatorname{im} B_u}{D_{\mathrm{all}}} \ge \frac{(1-\Delta_r-a)_+^2}{(1+\eta_1)\,(1+\eta_0+(1+a)^2 s/p)} - \delta_{\mathrm{row}}. \tag{12}$$

On the selected family the four small errors in (12) tend to zero and s is at most x^{-C_H} for a positive size exponent C_H. The estimate applies to every $\mathcal{S}$ at the same point, so in particular to arithmetic subsets of fixed positive dimension share, and more generally to sequences of subsets with $s/p \to 0$. The corresponding independent algebraic covectors can be chosen from actual source lattice vectors, because the lattice spans in each word. Evaluating on rows of RR produces a nonzero determinant of that order by surjectivity of the independent covectors onto L^{rank}. In the integral realization its determinant lies in the integer ring. The signed valuation bounds before the source shifts are the finite contribution supplied by Paper IV. Total column size · Row loss · Integral evaluation

The source uses a trace-square excess estimate as well. Put $Q = Q_{\mathcal{U}}$, and assume $0 < s \le 1/20$, $D_\star/M \le s, \eta_0, \eta_1, a \le s/2$, and $0 \le d_e \le 1/4$. By the same proof with the aggregate discarded proportion, $\operatorname{tr} Q \ge (1 - d_e - a)M$. The squared trace ratio therefore satisfies

$$\frac{D_{\mathrm{all}} \operatorname{tr} Q^2}{(\operatorname{tr} Q)^2} \le \frac{(1 + s/2)\,[1 + s/2 + s(1 + s/2)^2]}{(1 - s/2 - d_e)^2} \le 1 + 5s + 3\sqrt{d_e}. \tag{13}$$

For the second inequality at $d_e = 0$ the numerator is at most $1 + 2.1s$, and the denominator is at least $1 - s$, so the ratio is at most $1 + 5s$. For $z \in [0, 1/4]$, convexity bounds $((1 - s/2)/(1 - s/2 - z))^2$ by its secant between 0 and $1/4$. The slope of this secant is bounded by $4((39/29)^2 - 1) < 13/4$ since $s/2 \le 1/40$. Consequently the extra factor contributes at most $(1 + 5s)(13/4)d_e \le 3\sqrt{d_e}$. The aggregate diagonal defect is at most $e_0 M$, by compression of $K_{uu} - I$ on each word. Since $\sum r_u \ge (1 - d_e)M$, it is at most $2e_0 \sum r_u$, and $2e_0 \le 1/4$. This proves the frame estimates on the full ambient space after ramified source compression, with the arithmetic row loss still parametrized by δ_{row}. Source frame ratios and defect

Word weights and the lower spectral tail

Take a nonempty set S of n selected words and any nonnegative weights θ_u, and use the ramified columns $C'_u = A_u V_u$ of (11). Put $Q = \sum_{u \in S} \theta_u C'_u (C'_u)^*$ and $M_\theta = \sum_{u \in S} \theta_u d_u$. The diagonal bound $a^2 d_u$ in (10), with $a = \sqrt{e_0}$, gives

$$\operatorname{tr} Q \ge \sum_{u \in S} \theta_u (r_u - a\sqrt{r_u d_u}),$$

$$\operatorname{tr} Q^2 \le (1 + a)^2 \sum_{u \in S} \theta_u^2 d_u + \frac{1 + b^{-6}}{D_\star} \left(M_\theta^2 - \sum_{u \in S} \theta_u^2 d_u^2 \right).$$

Expanding the trace over ordered word pairs and compressing each block by the two source isometries gives these inequalities. One can keep the individual coefficient in (9) on each ordered pair. At unit weights $M_\theta = \sum_{u \in S} d_u$, so Cauchy–Schwarz gives $M_\theta^2 - \sum_{u \in S} d_u^2 \le (1 - 1/n) M_\theta^2$. The slope comparison below charges $\log |\mathcal{U}|$, with $\log |\mathcal{U}| \le (k - 3) \log 2$ in the window of Paper III. Trace and rank comparison · Word count and slope error

In dimension $D = D_{\mathrm{all}}$ with positive $\operatorname{tr} Q$, normalize all eigenvalues, including zeros, by $y_j = D\lambda_j(Q)/\operatorname{tr} Q$. With $\mathcal{E} = D \operatorname{tr} Q^2/(\operatorname{tr} Q)^2 - 1$, one has

$$\frac{1}{D} \sum_{j=1}^{D} (y_j - 1)^2 = \mathcal{E}, \qquad \frac{1}{D} \sum_j (1 - y_j)_+ = \frac{1}{D} \sum_j (y_j - 1)_+ \le \frac{\sqrt{\mathcal{E}}}{2}.$$

In particular the share of $y_j \le h < 1$ is at most $\mathcal{E}/(1-h)^2$, including the zero eigenvalues. Both signs of the deviation contribute to the moment. The span of eigenvectors above $h\ \mathrm{tr}\, Q/D$ intersects the common row subspace in codimension at most t_{row} within that span. The Rayleigh quotient on the intersection is still above the same threshold. Thus the compressed operator has at least $D - D\mathcal{E}/(1-h)^2 - t_{\mathrm{row}}$ eigenvalues above the threshold, by the min–max principle. Trace and rank comparison

6. Interface with Paper III and outstanding inputs

The finite lattice argument takes the common dual lattice, the maps B_u, their all-place height allowances and the mean, maximum and minimum slope hypotheses. The spectral contribution is (13) and the compressed diagonal defect. The source's joint ramified slope theorem takes these hypotheses, together with the identification of the arithmetic rows at one embedding and the row share $1 - \delta_{\mathrm{row}}$. If s_{lat} is the lower bound on the weighted mean of the adjusted source slopes, $u_{\mathrm{lat}} \ge 0$ bounds their upper spread, and $H_{\mathrm{lat}} \ge 0$ bounds s_{lat} minus the target minimum slope, it concludes

$$\mu(\mathrm{RR}^\vee) \ \ge\ s_{\mathrm{lat}} - \log|\mathcal{U}| - \frac{4u_{\mathrm{lat}}\log(1 + H_{\mathrm{lat}}/u_{\mathrm{lat}}) + (5s + 3\sqrt{d_e} + 4\sqrt{2e_0})H_{\mathrm{lat}}}{1 - \delta_{\mathrm{row}}}.$$

The continuous value of the first numerator term at $u_{\mathrm{lat}} = 0$ is zero. Paper III states the lattice hypotheses and uses the slope conclusion with the finite gains. Its filtered-lattice slope step is downstream of the rank and norm proofs in this paper. Numerical and lattice hypotheses · Joint ramified conclusion · Application to the common dual

For the ordinary translate mean of Paper III the surviving signed ramps are $G_{\mathbf{1}} + G_{\psi_C} - G_{\chi_C} - G_{\chi_C\psi_C}$, with the actual weights $w_\rho G_\xi(a_t/w_\rho)$ before thick-weight comparison. The local shifts of the integral columns use the signed term $\sum_j (g_{jw} + F_{jw} - E_{ujw})$ with the finite place weights $e_w \log N\mathfrak{p}_w/[K:\mathbb{Q}]$ in Paper III. The off-diagonal estimate applies at the same normalized family points for every word subset used there. Ordinary signed growth comparison · Ordinary weighted translate mean · Finite place comparison

For the ordinary selected family, (9) and (9a) now give the distinct-word entry of (10) under the quantitative hypotheses in §4. The diagonal defect, ambient row count, and ramified dimensions supply the other hypotheses for the downstream trace and slope estimates. Family assembly · Numerical hypotheses and assembly · Row and ramified dimensions

The remaining inputs for the distinct-word estimate are the typed expansion and its absolute convergence, the real disintegration and packet convergence in (5)–(7), and the weighted loop counts and amplitude bounds in (7b). The zero-Gram argument uses the pinned rational relevance and compact-type comparisons. Exact unfolding · Real convergence · Loop counts and amplitudes · Rational relevance · Compact-type comparison

For nongeneric corank one the remaining inputs are the class and quotient-measure identifications, the weighted local second moments and mass comparison, and the rotated weighted orbital estimate of Theorem B. For generic scalar corank one, (7f) proves the normalized nonsingular fibre mean, and (7g)–(7i) give the selected primitive densities and their uniform product bound. The finite signed comparison still uses the marked-corner lift, the signed partner sum and the orbital fibre mass identity with induced automorphism weights and matching norm factors, and the local class summability and restricted support. The mass comparison in (8) uses the selected completed-frame and Haar identifications, the typed level

indices, and the passage from the scalar row sum to the orbital signature. The block theta-kernel inequalities, real Poisson and packet estimates, and summability through the cutoff limit remain inputs. Nongeneric input assembly · Rotated orbital estimate · Marked-corner lift · Orbital fibre and partner identities · Selected volume transport · Scalar-row data · Poisson step and cutoff

The common lattice and Bessel comparison in §1, the equal-word estimates and row-dimension theorem, and the arithmetic height and slope hypotheses remain inputs for the rest of the paper. The lift-surjectivity argument in §1 still uses the full theta and doubling identities. Common pairing · Theta and doubling inputs · Diagonal and row count · Arithmetic height and slope · Lattice and matrix interface