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I. One-sided prime estimates on power intervals and the zeros of zeta

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Abstract. In Alpöge's argument, geometry bounds a centered sum of primes from above on intervals of the form $[Y, Y^{3/2}]$. The starts Y may be sparse. We prove the analytic implication for ordinary zeta and for positive sums of Dirichlet character ramps. A high derivative of the Mellin transform concentrates on a prescribed interval of logarithmic length. A single pole can then be detected even if the zeros have no largest real part. For ordinary zeta we identify the exponent of the largest positive fluctuation on *every* sufficiently late power interval, in the sense of a limit. At the critical exponent the possible limits of the interval maximum are an explicit constant and infinity. The arithmetic bound needed for RH is stated in §5. Alpöge's announcement · Released window theorem

1. What the prime estimate implies

Sections 2–4 reconstruct the localized criterion of Alpöge and Claude, and §6 gives proofs of two auxiliary estimates from their release. The additional conclusions of this paper are Theorem 2 and its endpoint corollary in §5. The recurrence argument there gives a limit, with no effective onset or rate of convergence. Published version and review

Let Λ be the von Mangoldt function, γ_E Euler's constant, and, for real $x \geq 1$, put

$$G(x) = \sum_{n \leq x} \Lambda(n)(x/n - 1) - x \log x + (1 + \gamma_E)x.$$

Write $G_+(x) = \max(G(x), 0)$. Fix $0 < \theta < 1$ and $\lambda > 1$.

Theorem 1. Suppose that for each $\eta > 0$ and each Y_0 there is a real $Y \geq \max(2, Y_0)$ such that

$$\int_Y^{Y^\lambda} G_+(x) x^{-\theta-1}\, dx \leq Y^\eta. \tag{1}$$

Then ζ has no zero in $\operatorname{Re} s > \theta$. In particular it suffices that there be a constant C and arbitrarily large Y for which $G(x) \leq Cx^\theta$ for every real $x \in [Y, Y^\lambda]$.

For the last assertion the integral is at most $\max(C, 0)(\lambda - 1) \log Y$. The source allows a nonnegative measurable majorant of G in (1). Its minimum possible value at each point is G_+. The same theorem holds

for a fixed finite positive sum of real parts of character ramps, with zero-freeness for each constituent, as shown in §4. Window and character definitions · Passage from a pointwise bound

2. Positive mass and a localized Mellin derivative

For $\operatorname{Re} s > 1$, absolute convergence and the substitution $x = nv$ give

$$\int_n^\infty (x/n - 1)x^{-s-1}\,dx = \frac{n^{-s}}{s(s-1)}.$$

Differentiating the Euler product and integrating the two centering terms consequently gives

$$\Phi(s) = \int_1^\infty G(x)x^{-s-1}\,dx = \frac{-\zeta'(s)/\zeta(s)}{s(s-1)} - \frac{1}{(s-1)^2} + \frac{1+\gamma_E}{s-1}. \tag{2}$$

At $s = 1$, $-\zeta'/\zeta = (s-1)^{-1} - \gamma_E + O(s-1)$. Thus both principal parts cancel. In $\operatorname{Re} s > 0$, the only poles of (2) are at the nontrivial zeros. A zero ρ of multiplicity m gives a simple pole with residue $m/[\rho(1-\rho)]$. This uses the classical nonvanishing on $\operatorname{Re} s = 1$. In particular the residue at a real pole in $(0,1)$ would be positive. Released Mellin and pole calculation · Classical zeta facts

Let $g(u) = G(e^u)$ for $u \geq 0$. The estimate $\sum_{n\leq x} \Lambda(n)/n \leq \sum_{2\leq n\leq x} (\log n)/n \ll 1 + (\log x)^2$ gives $|g(u)| \ll e^{2u}$. For $\sigma > 2$ and $t \in \mathbb{R}$ define $D_k(s) = (-1)^k \Phi^{(k)}(s)/k!$. Differentiation under the Laplace integral gives

$$|D_k(\sigma + it)| + \operatorname{Re} D_k(\sigma) \leq 2\int_0^\infty g_+(u)e^{-\sigma u}\frac{u^k}{k!}\,du. \tag{3}$$

Indeed, the absolute value is at most the integral with $|g|$, and the real-center term is the integral with g. The factor $|g| + g = 2g_+$ explains the one-sided hypothesis. Window inequality

Let $U = \log Y$, $a = \sigma - 2$, $h = (1+\lambda)/2$, and choose the positive integer $n = k+1$ nearest ahU. The probability density $a^n u^{n-1} e^{-au}/(n-1)!$ on $u > 0$ has mean n/a. As $U \to \infty$, the endpoints U, λU, divided by the mean, tend to $1/h$ and λ/h. Choose constants $1/h < r_- < 1 < r_+ < \lambda/h$. The probability outside this interval is at most $2e^{-cn}$ for all sufficiently large U, where $c = \min_{r\in\{r_-,r_+\}}(r - 1 - \log r) > 0$. To see this, the moment generating function of au is $(1-t)^{-n}$ for $t < 1$. Apply the exponential Markov bound at $t = 1 - 1/r$ to $au \leq rn$ for $r < 1$ and to $au \geq rn$ for $r > 1$. By the growth bound on g, the two parts outside $[U, \lambda U]$ in (3) together contribute $O(a^{-n}e^{-cn})$.

Fix σ large enough that $(\sigma-2)e^c > \sigma - \theta$. For **any fixed** $0 < b < \sigma - \theta$ we obtain the following conclusion along an unbounded sequence of k,

$$|D_k(\sigma + it)| + \operatorname{Re} D_k(\sigma) \leq C_b b^{-k} \quad (t \in \mathbb{R}). \tag{4}$$

For the middle interval, (1) becomes $\int_U^{\lambda U} g_+(u)e^{-\theta u}du \leq e^{\eta U}$. Since $v^k e^{-v}/k! \leq 1$ for $v \geq 0$, its contribution to (3) is at most $2e^{\eta U}(\sigma-\theta)^{-k}$. Choose $\eta > 0$ with $e^{\eta/(ah)} < (\sigma-\theta)/b$. Now take arbitrarily large good U for this η, observing $U/k \to 1/(ah)$. This proves (4), including the tails. For a pointwise majorant $Ce^{\theta u}$ the middle integral is bounded directly by $2\max(C,0)(\sigma-\theta)^{-k-1}$. High derivatives and tails · Released window argument

3. Detecting one pole

We give the argument for a meromorphic Φ in $\operatorname{Re} s > 0$ whose poles are discrete, simple, and in $0 < \operatorname{Re} s < 1$. Assume their residues are positive at real poles and (4) holds. Suppose some pole has real part $> \theta$. Write B for the supremum of the pole real parts, $R = \sigma - \theta$, and

$$d_0 = \min\bigl(R, \inf_{\rho} |\sigma - \rho|\bigr).$$

Then $d_0 \geq \sigma - B$ and $\sigma - B < R$. If $d_0 = \sigma - B$, the minimum is attained by discreteness on a compact disk of radius less than R. Equality in $|\sigma - \rho| \geq \sigma - \operatorname{Re}\rho$ makes $\rho = B$ the unique nearest pole. Take a circle of radius r strictly between d_0 and the next pole distance, with $r < R$. Subtracting its polar term leaves a holomorphic function on that closed disk. Cauchy's estimate yields

$$\operatorname{Re} D_k(\sigma) = c_B d_0^{-k-1} + O(r^{-k}), \qquad c_B > 0.$$

Choose $d_0 < b < r$ in (4) and set $t = 0$. This is impossible as $k \to \infty$.

If $d_0 > \sigma - B$, choose a pole ρ_* with $\sigma - \operatorname{Re}\rho_* < d_0$. For t close to $\operatorname{Im}\rho_*$ we still have $|\sigma + it - \rho_*| < d_0$. Choose such a t for which all pole distances from $s_* = \sigma + it$ are distinct. For two poles with different imaginary parts, equality of the squared distances is one linear equation with nonzero coefficient of t. For two with the same imaginary part their horizontal distances are different, since both lie to the left of σ. The poles form a countable set, so the excluded ordinates form a countable set. A nearest pole ρ_1 is attained within the closed disk of radius $|s_* - \rho_*| < R$ and is unique. Write $d = |s_* - \rho_1|$ and choose r strictly between d and the minimum of d_0 and the next pole distance at s_*. Both closed disks lie in $\operatorname{Re} s > 0$. Cauchy's estimate gives

$$|D_k(s_*)| \geq |c_1| d^{-k-1} - O(r^{-k}), \qquad |D_k(\sigma)| = O(r^{-k}).$$

Here c_1 is the nonzero residue. Taking $d < b < r$ contradicts (4) along its sequence. This proves Theorem 1, including the case where no zero has real part B. Nearest-pole dichotomy · Separation of the poles · Polar term and Cauchy estimate

4. Positive sums of characters

For a primitive nonprincipal Dirichlet character χ, replace the summand of the prime ramp by $\Lambda(n)\chi(n)(x/n - 1)$ and add $xL'(1,\chi)/L(1,\chi)$ to that sum. Denote it by $G_\chi(x)$. Its Mellin transform is

$$\Phi_\chi(s) = \frac{-L'(s,\chi)/L(s,\chi)}{s(s-1)} + \frac{L'(1,\chi)/L(1,\chi)}{s-1}.$$

The two residues at 1 cancel by the classical nonvanishing of $L(1,\chi)$. For the primitive character of conductor one take $G_{\mathbf{1}} = G$ and (2). For any finite family and positive weights w_j, let $G_F = \sum_j w_j \operatorname{Re} G_{\chi_j}$. Its transform is $\tfrac{1}{2}\sum_j w_j(\Phi_{\chi_j} + \Phi_{\bar\chi_j})$. The Dirichlet functional equations give meromorphic continuation, and the classical nonvanishing theorem on the whole line $\operatorname{Re} s = 1$ places the poles in the open strip. At any zero ρ of a constituent in the open critical strip the residue is

$$\frac{\sum_j w_j (m_{\chi_j}(\rho) + m_{\bar\chi_j}(\rho))}{2\rho(1-\rho)},$$

where m_χ is the zero multiplicity, taken to be zero at a regular nonzero value. The numerator is strictly positive. Thus every constituent zero survives in the family transform, even when several characters share it. Real poles have positive residues. The family is fixed while the derivative order varies, so its finite constants give the same growth estimate as in §2. Applying §§2–3 to G_F proves the character assertion. The extra Euler factors of an imprimitive inducing character have zeros only at $\operatorname{Re} s = 0$. The ordinary target in this series uses just G. Positive family and poles · Residue summation

5. The exact window exponent and the arithmetic premise

Let $B_\zeta = \sup_\rho \operatorname{Re}\rho$ over the nontrivial zeros. The classical zero count and functional equation give $1/2 \le B_\zeta \le 1$.

Theorem 2. For every fixed $\lambda > 1$,

$$\lim_{Y\to\infty} \max_{Y\le x\le Y^\lambda} \frac{\log(1+G_+(x))}{\log x} = B_\zeta. \tag{5}$$

Proof. For each $x \ge 2$, Mellin inversion of $(x/n-1)_+$ on $\operatorname{Re} s = 2$ applied to the uncentered ramp gives the integral of $-x^s(\zeta'/\zeta)(s)/[s(s-1)]$ with measure $ds/(2\pi i)$. Shift to $\operatorname{Re} s = -1$. For arbitrarily large symmetric horizontal heights T, the logarithmic derivative is $O(\log^2 T)$ across this fixed strip. Indeed the classical local partial-fraction estimate for ζ'/ζ has $O(\log T)$ terms within distance one in ordinate and an $O(\log T)$ remainder. The unit-interval zero count permits a height $\gg 1/\log T$ away from all these ordinates. The horizontal integrals tend to zero for each fixed x. On the left line the functional equation and the absolutely convergent logarithmic derivative at real part 2 give $|\zeta'/\zeta(-1+it)| \ll \log(2+|t|)$. Its integral is $O(x^{-1})$, with an absolute constant. At 1 the residue cancels the centering, and at 0 it is $\zeta'(0)/\zeta(0) = \log(2\pi)$. Counting multiplicities, the result is

$$G(x) = \log(2\pi) + \sum_\rho \frac{x^\rho}{\rho(1-\rho)} + O(x^{-1}).$$

The zero count $N(T) = O(T\log(2T))$ gives $\sum_\rho |\rho(1-\rho)|^{-1} < \infty$. Hence $|G(x)| \ll x^{B_\zeta}$ uniformly for $x \ge 2$, proving the upper limit in (5). If the lower limit were smaller, choose $0 < \theta < B_\zeta$ strictly between them. Along arbitrarily late intervals one would have $1 + G_+(x) \le x^\theta$ for every x there. Theorem 1 then excludes every zero to the right of θ, contrary to the definition of B_ζ. Continuity of the ramp at prime powers ensures that the displayed maximum exists. Classical contour estimates, Chapters II and IX · Zeta values at zero

Corollary (the endpoint constant). For $M_\lambda(Y) = \max_{Y\le x\le Y^\lambda} G(x)/\sqrt{x}$ and every fixed $\lambda > 1$,

$$\lim_{Y\to\infty} M_\lambda(Y) = \begin{cases} 2 + \gamma_E - \log(4\pi), & \text{if RH holds,} \\ +\infty, & \text{if RH fails.} \end{cases}$$

Under RH the explicit formula above gives $G(e^u)e^{-u/2} = H(u) + O(e^{-u/2})$, where $H(u) = \sum_\rho a_\rho e^{i\gamma_\rho u}$, $a_\rho = (1/4+\gamma_\rho^2)^{-1} > 0$, and $\gamma_\rho = \operatorname{Im}\rho$. Absolute summability gives uniform convergence. Conjugate pairs make H real, with $H(u) \le H(0)$. For any tolerance choose a finite set of frequencies whose omitted mass is smaller than that tolerance. The map $t \mapsto (e^{i\gamma_1 t}, \dots, e^{i\gamma_d t})$ has compact orbit closure in the torus. Cover it by finitely many translates of a small neighborhood of the identity centered at orbit times

$t_1, \dots, t_l$. For every real t, at least one $t - t_j$ then has all selected phases close to one. These returns occur in every interval of length $1 + \max_j t_j - \min_j t_j$. The finite sum approaches its value at zero at such returns, and the error from the tail is at most twice its mass. The intervals $[\log Y, \lambda \log Y]$ eventually contain a return for each fixed tolerance. This proves convergence of the maximum to $H(0)$.

For the completed zeta function $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$, subtract its Hadamard logarithmic derivatives at 1 and 0. The summands subtract to $1/[\rho(1-\rho)]$ and the sum converges absolutely. The functional equation makes the difference $2\xi'(1)/\xi(1)$. At 1, the expansion of $(s-1)\zeta(s)$ contributes γ_E, and $\Gamma'(1/2)/\Gamma(1/2) = -\gamma_E - 2\log 2$. Thus $H(0) = 2 + \gamma_E - \log(4\pi) = 0.0461914179\ldots$. Finally, if RH fails, fix $1/2 < \theta < B_\zeta$ by reflection. Equation (5) gives a point on every sufficiently late interval with $G(x) > x^\theta - 1$. Its normalized value is at least $Y^{\theta - 1/2} - Y^{-1/2}$, which tends to infinity. Hadamard product · Functional equation

Corollary (ordinary RH from the arithmetic estimate). Assume there is $\varepsilon_* > 0$ such that for each $0 < \varepsilon < \varepsilon_*$ there are arbitrarily late intervals $[Y, Y^{3/2}]$ on which, simultaneously at every real cutoff,

$$G(x) \le x^{1/2 + 9\varepsilon}. \tag{6}$$

Given a zero with $\beta = \operatorname{Re}\rho > 1/2$, choose $0 < \varepsilon < \min(\varepsilon_*, (\beta - 1/2)/9)$. Theorem 1 contradicts the existence of this zero. If $\beta < 1/2$, the functional equation gives a zero at $1 - \rho$, whose real part exceeds $1/2$. This proves the corollary. The family of geometric objects and the cutoff threshold in (6) may depend on ε. The function G itself is fixed. In the source the coefficient 9 is $e_D + e_L + 7$ with $e_D = 2, e_L = 0$. Papers II–VI trace the production of (6) and specify the geometric estimates still used as inputs in the prose. Production of the exponent · Application at a fixed level

6. Two estimates for the auxiliary characters

The character sums that occur before averaging in Paper II have conductors that grow with the cutoff. The following two estimates use information available *before* the determinant argument. Write N for the conductor of a primitive character χ. The classical local partial-fraction formula expresses $L'/L(s,\chi)$, with the polar term at one separated for zeta, as a sum of $1/(s-\rho)$ over zeros with $|\operatorname{Im}(s-\rho)| \le 1$, plus $O(\log(3N(2 + |\operatorname{Im} s|)))$, for $3/4 \le \operatorname{Re} s \le 2$. There are $O(\log(3N(2 + |\operatorname{Im} s|)))$ zeros in this sum, with multiplicity. These standard uniform forms follow by taking the logarithmic derivative of the completed Hadamard product, subtracting its value at $2 + i \operatorname{Im} s$, and bounding the distant terms by the local zero count. We also use the classical zero-free region $\operatorname{Re} s \ge 1 - c/\log(3N(2 + |\operatorname{Im} s|))$ with its possible real exceptional zero for a real nonprincipal character. Davenport, *Multiplicative Number Theory*, Chapters 12–14

Proposition 3 (finite sum on a fixed-width zero-free rectangle). Fix $A \ge 1$ and $0 < \tau \le 1$. For Y sufficiently large put $X = Y^A$, $Z = X^{400}$. Suppose a primitive nonprincipal χ has conductor $N \le Q$, with $Y^{\tau/6} \le \log Q \le Y^{\tau/4}$. Suppose $L(s,\chi)$ has no zero with $\operatorname{Re} s > 1 - 1/400$ and $|\operatorname{Im} s| \le Q$. Then

$$X \left| -\frac{L'(1,\chi)}{L(1,\chi)} - \sum_{n \le Z} \frac{\Lambda(n)\chi(n)}{n} \right| \le Y^{\tau}. \tag{7}$$

Proof. Set $z = \lfloor Z \rfloor + 1/2$, $T = z^3$, $b = 1/\log z$ and $F(w) = -(L'/L)(1 + w, \chi)$. The sum up to Z equals the sum up to z. Truncated Perron inversion integrates $F(w) z^w / w$ over $[b - iT, b + iT]$ with

measure $dw/(2\pi i)$. Its error is bounded by a constant times

$$\sum_{n\ge 2}\frac{\log n}{n}(z/n)^b\min\{1,(T|\log(z/n)|)^{-1}\}\ll\frac{(\log z)^2}{T}.$$

Outside $[z/2,2z]$ use $\sum_{n\ge 2}(\log n)/n^{1+b}\ll b^{-2}$. Inside this interval $|\log(z/n)|\gg|z-n|/z$ and $\sum_{z/2\le n\le 2z}|z-n|^{-1}\ll\log z$ by the half-integer choice. Take $\kappa=\tau/(1600A)$ and $\alpha=1/400-\kappa$. For all sufficiently large Y, $T+2<Q$. Throughout the rectangle $-\alpha\le\operatorname{Re}w\le b$, $|\operatorname{Im}w|\le T$, every zero contributing to the local partial-fraction formula for F is at horizontal distance at least κ. Thus $|F|\ll_\kappa\log(3Q(T+3))\ll_\kappa\log Q$. The only pole crossed on shifting the integral is that of $1/w$, with residue $F(0)$. The two horizontals contribute $O_\kappa((\log Q)/T)$ and the left vertical contributes $O_\kappa(z^{-\alpha}\log Q\log T)$. Since $Xz^{-\alpha}\ll Y^{\tau/4}$ and $\log Q\le Y^{\tau/4}$, the left vertical times X is $O_{A,\tau}(Y^{\tau/2}\log Y)$. The errors involving X/T tend to zero. This proves (7), including the unit constant after increasing the onset. Released rectangle estimate and cutoff

For $Y\le y\le X$ define $S(y)=\sum_{n\le Z}\Lambda(n)\chi(n)\min(1,y/n)$. The identity $G_\chi(y)=-S(y)+y[\ell_\chi+\sum_{n\le Z}\Lambda(n)\chi(n)/n]$, with $\ell_\chi=L'(1,\chi)/L(1,\chi)$, follows by splitting at $n=y$. Thus (7) is exactly the error paid when the progression moment in Paper II estimates $S(y)$. If $|G_\chi(Y)|\le Y^{1/2+\varepsilon/2}$ with $0<\varepsilon<1$, then $|\ell_\chi|\le Y^{-1/2+\varepsilon/2}+\sum_{n\le Y}\Lambda(n)/n\le 4\log Y$ eventually, by partial summation and Chebyshev's bound $\sum_{n\le u}\Lambda(n)\le 3u$. This last estimate applies uniformly for the selected characters.

Proposition 4 (the uniform bound at the larger cutoff). Fix positive A,ε,c_0,c_6. Suppose $\log N\le Ax^{\varepsilon/5}(\log x)^2$. If χ is real nonprincipal, assume $L(s,\chi)$ has no real zero in $(1-c_0,1)$. At conductor one use $G_{\mathbf 1}=G$. Uniformly in the character and in every real $y\ge e^{x^{\varepsilon/3}}/3$, for all sufficiently large x,

$$|G_\chi(y)|\le ye^{-c_6x^{\varepsilon/8}}.\tag{8}$$

Proof. Write $V=\log y$, $T=e^{\sqrt V}$ and $L_0=\log(3N(T+3))$. Choose an absolute constant $c'>0$ small enough for the zero-free region, and set $\delta=c'/L_0$. Once x is sufficiently large, $2\delta<c_0$. There are no zeros with $\operatorname{Re}s\ge 1-2\delta$ and $|\operatorname{Im}s|\le T+1$. Use the Mellin integral of the uncentered ramp on $\operatorname{Re}s=1+1/V$, with kernel $y^s/[s(s-1)]$. The portion with $|\operatorname{Im}s|>T$ is $O(yV/T)$ by absolute convergence and $-\zeta'/\zeta(1+1/V)\ll V$. Move the truncated contour to $\operatorname{Re}s=1-\delta$. On the new vertical and on both horizontals the local partial-fraction bound gives $|L'/L|\ll L_0^2$, also allowing the polar term of zeta. At 1 the residue for nonprincipal χ is $-y\ell_\chi$. At conductor one it is $y(\log y-1-\gamma_E)$. The centering cancels these residues exactly. Bounding the integral on $|t|\le 1$ of the new vertical kernel by $O(\delta^{-1})$ and using the integrable t^{-2} bound elsewhere gives

$$|G_\chi(y)|\ll y\left(e^{-\delta V}L_0^3+L_0^2/T^2+V/T\right).$$

Here $V\ge x^{\varepsilon/3}-\log 3$, so $L_0\ll_{A,\varepsilon}V^{3/5}(\log V)^2$ for every such y. Consequently $\delta V\gg_{A,\varepsilon}V^{2/5}/(\log V)^2$. Both this lower bound and $\sqrt V$ divided by $V^{3/8}$ tend to infinity. Since $x^{\varepsilon/8}\le(V+\log 3)^{3/8}$, all three terms fit in (8) for the prescribed c_6. Uniform estimate in the release · Application to the host conductors

Apply these estimates to the primitive Dirichlet representatives of the host characters in Paper II. Every real character has the form $\xi J_q^aJ_d^b$ on units, with $a,b\in\{0,1\}$ and ξ of conductor dividing 8. Indeed a

quadratic character on the dyadic units is trivial on their squares, which contain every class congruent to 1 modulo 8. When $a = b = 0$, classical nonvanishing at 1 and finiteness give a common $c_0 > 0$ for the nonprincipal choices. Otherwise the selected rectangle in Paper II contains the real interval $(399/400, 1)$. Taking the smaller of the two widths makes Proposition 4 applicable to the entire host. The cutoff in (8) is the one used for partners assigned the smaller weight b' in Paper II. Conductor splitting and the real interval